6.5 三重積分

習題解答 **6.5.3.**

(1)

$$
\begin{aligned}
& \iiint_\Omega x + y^2 + z^3 \; dV \\
= \; & \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 x + y^2 + z^3 \; dz \; dy \; dx \\
= \; & \int_{-1}^1 \int_{-1}^1 \left(xz + y^2 z + \frac{z^4}{4} \right)\Big|_{-1}^1 \; dy \; dx \\
= \; & \int_{-1}^1 \int_{-1}^1 2x + 2y^2 \; dy \; dx \\
= \; & 2\int_{-1}^1 \left(xy + \frac{y^3}{3} \right)\Big|_{-1}^1 \; dx \\
= \; & 2\int_{-1}^1 2x + \frac{2}{3} \; dx = 4\Big(\frac{x^2}{2} + \frac{1}{3}x\Big)\Big|_{-1}^1 = \frac{8}{3}
\end{aligned}
$$

(2)

$$
\begin{aligned}
\iiint_\Omega xyz \; dV &= \int_0^1 \int_0^2 \int_0^3 xyz \; dz \; dy \; dx \\
&= \int_0^1 x \; dx \cdot \int_0^2 y \; dy \cdot \int_0^3 z \; dz = \frac{1}{2} \cdot \frac{4}{2} \cdot \frac{9}{2} = \frac{9}{2}
\end{aligned}
$$

(3)

$$
\begin{aligned}
& \iiint_\Omega x \; dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x \; dz \; dy \; dx \\
= \; & \int_0^1 \int_0^{1-x} x(1-x-y) \; dy \; dx = \int_0^1 \left((x-x^2)y - x\frac{y^2}{2} \right)\Big|_0^{1-x} \; dx \\
= \; & \int_0^1 \frac{x(1-x)^2}{2} \; dx = \frac{1}{2}\int_0^1 x - 2x^2 + x^3 \; dx \\
= \; & \frac{1}{2}\Big(\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\Big)\Big|_0^1 = \frac{1}{2}\Big(\frac{1}{2} - \frac{2}{3} + \frac{1}{4}\Big) = \frac{1}{24}
\end{aligned}
$$

(4)

$$
\begin{aligned}
\iiint_\Omega ye^z \; dV &= \int_0^1 \int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} ye^z \; dy \; dx \; dz \\
&= \int_0^1 \int_{-1}^1 e^z \Big(\frac{y^2}{2}\Big)\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \; dx \; dz = 0
\end{aligned}
$$

習題解答 6.5.4.

(1) 調換 x 和 y 的順序：

$$\begin{aligned}
& \int_0^1\int_0^1\int_{x^2}^1 xze^{zy^2}\,dy\,dx\,dz = \int_0^1\int_0^1\int_0^{\sqrt{y}} xze^{zy^2}\,dx\,dy\,dz \\
= & \int_0^1\int_0^1 ze^{zy^2}\Big(\frac{x^2}{2}\Big)\Big|_0^{\sqrt{y}}\,dy\,dz = \frac{1}{2}\int_0^1\int_0^1 zye^{zy^2}\,dy\,dz \\
= & \frac{1}{2}\int_0^1\Big(\frac{e^{zy^2}}{2}\Big)\Big|_0^1\,dz = \frac{1}{4}\int_0^1 e^z-1\,dz \\
= & \frac{1}{4}(e^z-z)\big|_0^1 = \frac{1}{4}(e-1-1) = \frac{e-2}{4}
\end{aligned}$$

(2) 調換 x 和 y 的順序：

$$\begin{aligned}
& \int_0^4\int_0^1\int_{2y}^2 \frac{\cos x^2}{\sqrt{z}}\,dx\,dy\,dz = \int_0^4\int_0^2\int_0^{\frac{x}{2}} \frac{\cos x^2}{\sqrt{z}}\,dy\,dx\,dz \\
= & \int_0^4\int_0^2 \frac{x\cos x^2}{2\sqrt{z}}\,dx\,dz = \frac{1}{4}\int_0^4\frac{1}{\sqrt{z}}\,dz\cdot\int_0^2 x\cos x^2\,dx \\
= & \frac{1}{2}\Big(2z^{\frac{1}{2}}\Big)\Big|_0^4\cdot\Big(\frac{\sin x^2}{2}\Big)\Big|_0^2 = \frac{1}{2}\cdot 4\cdot\frac{\sin 4}{2} = \sin 4
\end{aligned}$$

習題解答 6.5.5.

$$\begin{aligned}
\text{體積} &= \int_0^a\int_0^{b(1-\frac{x}{a})}\int_0^{c(1-\frac{x}{a}-\frac{y}{b})} 1\,dz\,dy\,dx \\
&= \int_0^a\int_0^{b(1-\frac{x}{a})} c(1-\frac{x}{a}-\frac{y}{b})\,dy\,dx \\
&= c\int_0^a\Big(y(1-\frac{x}{a})-\frac{y^2}{2b}\Big)\Big|_0^{b(1-\frac{x}{a})}\,dx \\
&= c\int_0^a b(1-\frac{x}{a})-\frac{b^2(1-\frac{x}{a})^2}{2b}\,dx = \frac{bc}{2}\int_0^a(1-\frac{x}{a})^2\,dx \\
&= \frac{bc}{2}\cdot\Big((-a)\frac{(1-\frac{x}{a})^3}{3}\Big)\Big|_0^a = \frac{bc}{2}\cdot\frac{a}{3} = \frac{abc}{6}
\end{aligned}$$

習題解答 6.5.11.

本題變數變換取法和上題一樣, $J(u,v,w)=abc$. 故積分為

$$\iiint_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1} x^2y^2\,dx\,dy\,dz = \iiint_{u^2+v^2+w^2\le 1} a^2b^2u^2v^2\cdot abc\,du\,dv\,dw$$

現再用球面坐標來計算, 得原式為

$$\begin{aligned}
\text{原式} &= a^3b^3c\int_0^{2\pi}\int_0^{\pi}\int_0^1(\rho\sin\phi\cos\theta)^2(\rho\sin\phi\sin\theta)^2\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= a^3b^3c\int_0^{2\pi}(\cos\theta\sin\theta)^2\,d\theta\cdot\int_0^{\pi}\sin^5\phi\,d\phi\cdot\int_0^1\rho^6\,d\rho \\
&= a^3b^3c\int_0^{2\pi}\frac{1-\cos 4\theta}{8}\,d\theta\cdot\int_0^{\pi}-(1-\cos^2\phi)^2\,d(\cos\phi)\cdot\Big(\frac{\rho^7}{7}\Big)\Big|_0^1 \\
&= \frac{a^3b^3c}{7}\Big(\frac{\theta}{8}-\frac{\sin 4\theta}{32}\Big)\Big|_0^{2\pi}\cdot\Big(-\cos\theta+\frac{2\cos^3\theta}{3}-\frac{\cos^5\theta}{5}\Big)\Big|_0^{\pi} \\
&= \frac{a^3b^3c}{7}\cdot\frac{\pi}{4}\cdot 2\cdot(1-\frac{2}{3}+\frac{1}{5}) = \frac{4}{105}a^3b^3c\,\pi
\end{aligned}$$

習題解答 **6.5.12.**

依題意取坐標變換如下

$$\begin{cases} u = x + y \\ v = x - y \\ w = \ln z \end{cases} \quad \Leftrightarrow \quad \begin{cases} x = \frac{u+v}{2} \\ y = \frac{u-v}{2} \\ z = e^w \end{cases}$$

區域邊界的曲線對應如下：

$$\begin{cases} y = x - 1, y = x + 2 \\ y = -x, \ \ y = -x + 1 \\ z = 1, \ \ z = 2 \end{cases} \quad \Leftrightarrow \quad \begin{cases} v = 1, \ \ v = -2 \\ u = 0, \ \ u = 1 \\ w = 0, \ \ w = \ln 2 \end{cases}$$

再計算 $J(u, v, w)$ 如下：

$$J(u, v, w) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial x}{\partial w} & \dfrac{\partial y}{\partial w} & \dfrac{\partial z}{\partial w} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2} & \dfrac{1}{2} & 0 \\ \dfrac{1}{2} & -\dfrac{1}{2} & 0 \\ 0 & 0 & e^w \end{vmatrix} = -\frac{e^w}{2}$$

原積分經變數變換後得

$$\begin{aligned} \int_0^1 \int_{-2}^1 \int_0^{\ln 2} \frac{uvw}{e^w} \cdot \frac{e^w}{2} \, dw \, dv \, du &= \frac{1}{2} \int_0^1 u \, du \cdot \int_{-2}^1 v \, dv \cdot \int_0^{\ln 2} w \, dw \\ &= \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1-4}{2} \cdot \frac{(\ln 2)^2}{2} = -\frac{3(\ln 2)^2}{16} \end{aligned}$$