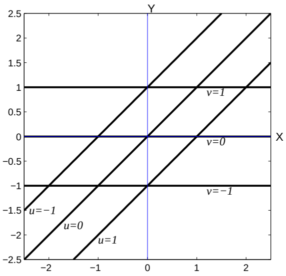
6.4 二重積分之變數變換

習題解答 **6.4.1.**

(1)

$$\begin{cases} x = u+v \\ y = v \end{cases} \Leftrightarrow \begin{cases} u = x-y \\ v = y \end{cases}$$

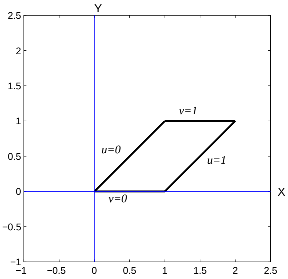
所以 $u = C$ 對應到 $x - y = C$; $v = C$ 對應到 $y = C$. 答案如右上圖.



(2) 由 (1), $u + v = 1$ 對應到 $x = 1$.

(3) 由 (1), $x^2+y^2 = 1$ 對應到 $(u+v)^2+v^2 = 1$, 展開得 $u^2 + 2uv + 2v^2 = 1$. (這是一個斜橢圓)

(4) 由前對應可得: $0 \le u \le 1,\ 0 \le v \le 1$ 對應到 $0 \le x - y \le 1$, $0 \le y \le 1$ (如右下圖).



習題解答 **6.4.4.**

取坐標變換如下

$$\begin{cases} u = xy \\ v = \frac{y}{x} \end{cases} \Leftrightarrow \begin{cases} x = \sqrt{uv} \\ y = \sqrt{\frac{u}{v}} \end{cases}$$

區域邊界的曲線對應如下：

$$\begin{cases} xy = 1, \quad xy = 9 \\ y = x, \quad y = 2x \end{cases} \Leftrightarrow \begin{cases} u = 1, \quad u = 9 \\ v = 1, \quad v = 2 \end{cases}$$

再計算 $J(u, v)$ 如下：

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{2\sqrt{uv}} & \dfrac{\sqrt{v}}{2\sqrt{u}} \\ -\dfrac{\sqrt{u}}{2v\sqrt{v}} & \dfrac{\sqrt{u}}{2\sqrt{v}} \end{vmatrix} = \frac{1}{4v} + \frac{1}{4v} = \frac{1}{2v}$$

原積分經變數變換後得

$$\begin{aligned} & \int_1^9 \int_1^2 (\sqrt{u} + \sqrt{2v}) \cdot \frac{1}{2v}\ dv\ du = \int_1^9 \frac{\sqrt{u}}{2} \ln v\Big|_1^2 + \sqrt{2v}\Big|_1^2\ du \\ = & \int_1^9 \frac{\ln 2}{2}\sqrt{u} + (2 - \sqrt{2})\ du = \frac{\ln 2}{2} \cdot \frac{2}{3} u^{\frac{3}{2}}\Big|_1^9 + (2 - \sqrt{2})(9 - 1) \\ = & \frac{26}{3} \ln 2 + 16 - 8\sqrt{2} \end{aligned}$$

$\boxed{\text{習題解答}}$ **6.4.6.**

原區域為 $1 \leq u \leq 2$, $2u-2 \leq v \leq u$. 依題意取坐標變換如下

$$\begin{cases} x = v - u + 1 \\ y = v - 2u \end{cases} \quad \Leftrightarrow \quad \begin{cases} u = x - y - 1 \\ v = 2x - y - 2 \end{cases}$$

區域邊界的曲線對應如下：

$$\begin{cases} u = 1, \ \ v = 2u - 2 \\ v = u \end{cases} \quad \Leftrightarrow \quad \begin{cases} y = x - 2, \ \ y = -2 \\ x = 1 \end{cases}$$

再計算 $J(x, y)$ 如下：

$$J(x,y) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ -1 & -1 \end{vmatrix} = 1$$

原積分經變數變換後得

$$\int_0^1 \int_{-2}^{x-2} e^{x^2} \cdot 1 \, dy \, dx = \int_0^1 x e^{x^2} \, du = \frac{e^{x^2}}{2}\Big|_0^1 = \frac{e-1}{2}$$

$\boxed{\text{習題解答}}$ **6.4.7.**

依題意, 取坐標變換如下

$$\begin{cases} x = u - 2v \\ y = 3u + v \end{cases} \quad \Leftrightarrow \quad \begin{cases} u = \dfrac{2y + x}{7} \\ v = \dfrac{y - 3x}{7} \end{cases}$$

區域邊界的曲線對應如下：

$$\begin{cases} 2u + 3v = 0, \ \ 2u + v = 0 \\ u - 2v = 1 \end{cases} \quad \Leftrightarrow \quad \begin{cases} y = x, \ \ y = 0 \\ x = 1 \end{cases}$$

再計算 $J(u, v)$ 如下：

$$J(x,y) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{1}{7} & -\dfrac{3}{7} \\ \dfrac{2}{7} & \dfrac{1}{7} \end{vmatrix} = \frac{1}{7}$$

原積分經變數變換後得

$$\begin{aligned} \int_0^1 \int_0^x (\sqrt{x} + \sqrt{y}) \cdot \frac{1}{7} \, dv \, dx &= \frac{1}{7}\int_0^1 x^{\frac{3}{2}} + (\frac{2y^{\frac{3}{2}}}{3})\Big|_0^x \, dx \\ &= \frac{1}{7}\int_0^1 \frac{5}{3} x^{\frac{3}{2}} \, dx = \frac{1}{7} \cdot \frac{5}{3} \cdot \frac{2}{5} x^{\frac{5}{2}}\Big|_0^1 = \frac{2}{21} \end{aligned}$$

習題解答 6.4.9.

依題意取坐標變換如下

$$\begin{cases} u = y - x \\ v = 2x + y \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{v-u}{3} \\ u = \dfrac{2u+v}{3} \end{cases}$$

區域邊界的曲線對應如下：

$$\begin{cases} y - x = 1, \;\; y - x = 2 \\ 2x + y = 0, \;\; 2x + y = 2 \end{cases} \Leftrightarrow \begin{cases} u = 1, \;\; u = 2 \\ v = 0, \;\; v = 2 \end{cases}$$

再計算 $J(u,v)$ 如下：

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{1}{3} & \dfrac{2}{3} \\ \dfrac{1}{3} & \dfrac{1}{3} \end{vmatrix} = -\frac{1}{3}$$

原積分經變數變換後得

$$\int_1^2 \int_0^2 1 \cdot \big| -\frac{1}{3} \big| \, dv \, du = \frac{1}{3} \cdot (2-1) \cdot (2-0) = \frac{2}{3}$$

習題解答 6.4.10.

依題意取坐標變換如下

$$\begin{cases} u = a_1 x + b_1 y \\ v = a_2 x + b_2 y \end{cases} \Leftrightarrow \begin{cases} x = \dfrac{b_2 u - b_1 v}{\Delta} \\ u = \dfrac{-a_2 u + a_1 v}{\Delta} \end{cases}$$

其中 $\Delta = a_1 b_2 - a_2 b_1$. 而區域邊界的曲線對應如下：

$$\begin{cases} a_1 x + b_1 y = c_{11}, \;\; a_1 x + b_1 y = c_{12} \\ a_2 x + b_2 y = c_{21}, \;\; a_2 x + b_2 y = c_{22} \end{cases} \Leftrightarrow \begin{cases} u = c_{11}, \;\; u = c_{12} \\ v = c_{21}, \;\; v = c_{22} \end{cases}$$

再計算 $J(u,v)$ 如下：

$$J(u,v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\dfrac{b_2}{\Delta} & \dfrac{-a_2}{\Delta} \\ \dfrac{-b_1}{\Delta} & \dfrac{a_1}{\Delta} \end{vmatrix} = \frac{a_1 b_2 - a_2 b_1}{\Delta^2} = \frac{1}{\Delta}$$

原積分經變數變換後得

$$\int_{c_{11}}^{c_{12}} \int_{c_{21}}^{c_{22}} 1 \cdot \big| \frac{1}{\Delta} \big| \, dv \, du = \frac{1}{|\Delta|} \cdot (c_{12} - c_{11}) \cdot (c_{21} - c_{22})$$

$\boxed{\text{習題解答}}$ **6.4.11.**

(1) 用極坐標得,

$$x^2+y^2 \le R^2 \quad \Leftrightarrow \quad 0 \le \theta \le 2\pi,\ 0 \le r \le R$$

則原積分變為

$$\int_0^{2\pi}\int_0^R r^2(cos^2\theta - \sin^2\theta)\cdot r\ dr\ d\theta = \int_0^{2\pi} \cos 2\theta\ d\theta \cdot \int_0^R r^3\ dr$$

但

$$\int_0^{2\pi} \cos 2\theta\ d\theta = \frac{\sin 2\theta}{2}\Big|_0^{2\pi} = 0$$

所以

$$\iint_{x^2+y^2\le R^2} x^2-y^2\ dA = 0 = (\pi R^2)\cdot f(0,0)$$

(2) 用類似極坐標的坐標變換:

$$x = 1 + r\cos\theta,\ y = r\sin\theta$$

則積分範圍對應如下

$$(x-1)^2+y^2 \le R^2 \quad \Leftrightarrow \quad 0 \le \theta \le 2\pi,\ 0 \le r \le R$$

再計算 $J(r,\theta)$ 如下：

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

原積分變為

$$\begin{aligned} & \int_0^{2\pi}\int_0^R \big((1+r\cos\theta)^2 - (r\sin\theta)^2\big)\cdot r\ dr\ d\theta \\ = & \int_0^{2\pi}\int_0^R \big(1+2r\cos\theta + r^2\cos 2\theta\big)\cdot r\ dr\ d\theta \\ = & \int_0^{2\pi}\int_0^R r\ dr\ d\theta \qquad (\cos\theta,\ \cos 2\theta \text{ 對 } \theta \text{ 積分為 } 0) \\ = & \int_0^{2\pi} d\theta \cdot \int_0^R r\ dr = \pi R^2 = \pi R^2 \cdot f(1,0) \end{aligned}$$

(3) 用類似極坐標的坐標變換:

$$x = \alpha + r\cos\theta,\ y = \beta + r\sin\theta$$

則積分範圍對應如下

$$(x-\alpha)^2+(y-\beta)^2 \le R^2 \quad \Leftrightarrow \quad 0 \le \theta \le 2\pi,\ 0 \le r \le R$$

再計算 $J(r,\theta)$ 如下：

$$J(r,\theta) = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial y}{\partial r} \\ \dfrac{\partial x}{\partial \theta} & \dfrac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r$$

原積分變為

$$\begin{aligned}
& \int_0^{2\pi}\int_0^R \big((\alpha+r\cos\theta)^2-(\beta+r\sin\theta)^2\big)\cdot r\ dr\ d\theta \\
= & \int_0^{2\pi}\int_0^R \big(\alpha^2-\beta^2+2\alpha r\cos\theta-2\beta r\sin\theta+r^2\cos 2\theta\big)\cdot r\ dr\ d\theta \\
= & (\alpha^2-\beta^2)\int_0^{2\pi}\int_0^R r\ dr\ d\theta \qquad (\cos\theta,\ \sin\theta,\ \cos 2\theta \text{ 對 } \theta \text{ 積分為 } 0) \\
= & (\alpha^2-\beta^2)\int_0^{2\pi} d\theta\cdot\int_0^R r\ dr = \pi R^2(\alpha^2-\beta^2) = \pi R^2\cdot f(\alpha,\beta)
\end{aligned}$$