6.3 二重積分的極座標形式

習題解答 **6.3.2.**

(1) $4\cos\frac{\pi}{3}=2,\quad \sin\frac{\pi}{3}=2\sqrt{3} \quad\Rightarrow\quad [4,\frac{\pi}{3}]=(2,2\sqrt{3})$

(2) $[-1,\frac{5\pi}{4}]=[1,\frac{5\pi}{4}-\pi]=[1,\frac{\pi}{4}]=(\cos\frac{\pi}{4},\sin\frac{\pi}{4})=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

或

$[-1,\frac{5\pi}{4}]=((-1)\cos\frac{\pi}{5},(-1)\sin\frac{\pi}{5})=((-1)\cdot\frac{1}{\sqrt{2}},(-1)\cdot\frac{1}{\sqrt{2}})=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$

(3) $[0,2\pi]=(0,0)$

(4) $(1,0)=[1,0+2n\pi]=[1,2n\pi]$ 或 $[-1,\pi+2n\pi]$, 其中 $n\in\mathbb{Z}$.

(5) 先找出最熟悉的解 $(0,1)=[1,\frac{\pi}{2}]$, 再寫出一般解 $(1,0)=[1,\frac{\pi}{2}+2n\pi]$ 或 $[-1,\frac{\pi}{2}+2n\pi]$, 其中 $n\in\mathbb{Z}$.

(6) $r=\sqrt{(-1)^2+(-1)^2}=\sqrt{2}$, $\tan\theta=\frac{-1}{-1}=1$, 但點在第三象限, 取角度為 $\frac{3\pi}{4}$.

先找出最熟悉的解 $(-1,-1)=[\sqrt{2},\frac{3\pi}{4}]$, 再寫出一般解 $(-1,-1)=[\sqrt{2},\frac{3\pi}{4}+2n\pi]$ 或 $[-\sqrt{2},(\frac{3\pi}{4}-\frac{\pi}{2})+2n\pi]=[-\sqrt{2},\frac{\pi}{4}+2n\pi]$, 其中 $n\in\mathbb{Z}$.

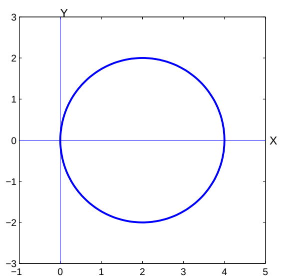
習題解答 **6.3.3.**

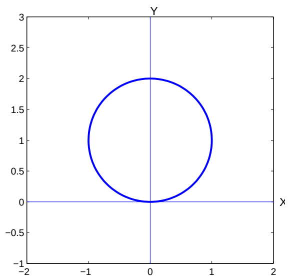
(1) $r=4\cos\theta \;\Rightarrow\; r^2=4r\cos\theta\Rightarrow x^2+y^2=4x \;\Rightarrow\; (x-2)^2+y^2=4$. (下頁左圖)

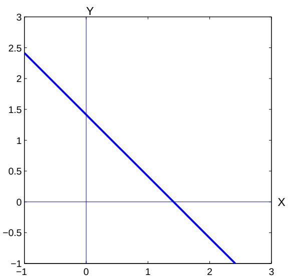
(2) $r=2\sin\theta \;\Rightarrow\; r^2=2r\sin\theta\Rightarrow x^2+y^2=2y \;\Rightarrow\; x^2+(y-1)^2=1$. (下頁中圖)

(3) (下頁右圖)

$$\begin{aligned} r=4\cos(\theta-\frac{\pi}{4})=1 \quad&\Rightarrow\quad r(\cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4})=1\\ &\Rightarrow\quad \frac{1}{\sqrt{2}}r\cos\theta+\frac{1}{\sqrt{2}}=1 \quad\Rightarrow\quad x+y=\sqrt{2}\end{aligned}$$







習題解答 **6.3.4.**

(1) $f(x,y)=x^2-y^2=r^2(\cos^2\theta-\sin^2\theta)=r^2\cos 2\theta$

(2) $f(x,y)=xy=r^2\cos\theta\sin\theta=\frac{1}{2}r^2\sin 2\theta$

(3) $f(x,y)=x+y-1=r\cos\theta+r\sin\theta-1$

(4) $f(x,y)=e^{-(x^2+y^2)}=e^{-r^2(\cos^2\theta+\sin^2\theta)}=e^{-r^2}$

習題解答 **6.3.5.**

(1)

$$\begin{aligned}\iint_\Omega f[r,\theta]\, dA &= \int_0^\pi \left(\int_2^4 r\sin r\cdot r\; dr\right) d\theta = \int_0^\pi \frac{1}{3}\sin\theta \left(r^3\big|_2^4\right) d\theta \\ &= \frac{56}{3}\int_0^\pi \sin\theta\; d\theta = -\frac{56}{3}\cos\theta\big|_0^\pi = \frac{112}{3}\end{aligned}$$

(2)

$$\begin{aligned}\iint_\Omega f[r,\theta]\, dA &= \int_0^\pi \left(\int_0^3 \left(\sqrt{1+2r^2}+\theta\right) r\; dr\right) d\theta \\ &= \int_0^\pi \left(\frac{1}{6}(1+2r^2)^{\frac{3}{2}} + \frac{1}{2}\theta r^2\bigg|_0^3\right) d\theta \\ &= \int_0^\pi \frac{1}{6}(19^{\frac{3}{2}}-1) + \frac{9}{2}\theta\; d\theta = \frac{1}{6}(19^{\frac{3}{2}}-1)\pi + \frac{9}{4}\pi^2\end{aligned}$$

(3)

$$\begin{aligned}&\int_0^{\frac{\pi}{4}}\int_{\sin\theta}^{\cos\theta} \sqrt{1-r^2}\tan\theta\cdot r\; dr\; d\theta = \int_0^{\frac{\pi}{4}} -\frac{\tan\theta}{3}(1-r^2)^{\frac{3}{2}}\big|_{\sin\theta}^{\cos\theta}\; d\theta \\ = \; & -\frac{1}{3}\int_0^{\frac{\pi}{4}} \tan\theta(\sin^3\theta-\cos^3\theta)\; d\theta = -\frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\sin^4\theta}{\cos\theta} - \sin\theta\cos^2\theta)\; d\theta\end{aligned}$$

分開計算

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \frac{\sin^4\theta}{\cos\theta}\; d\theta &= \int_0^{\frac{\pi}{4}} \frac{\sin^4\theta}{\cos^2\theta}\cdot\cos\theta\; d\theta \overset{u=\sin\theta}{=} \int_0^{\frac{1}{\sqrt{2}}} \frac{u^4}{1-u^2}\; du \\ &= \int_0^{\frac{1}{\sqrt{2}}} -(u^2+1) + \frac{1}{1-u^2}\; du \\ &= \left(-\frac{u^3}{3} - u + \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right|\right)\Big|_0^{\frac{1}{\sqrt{2}}} \\ &= -\frac{1}{6\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{1}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1} = -\frac{7\sqrt{2}}{12} + \ln(\sqrt{2}+1) \\ \int_0^{\frac{\pi}{4}} \sin\theta\cos^2\theta)\; d\theta &= -\frac{\cos^3\theta}{3}\Big|_0^{\frac{\pi}{4}} = -\frac{\frac{1}{2\sqrt{2}}}{3} + \frac{1}{3} = \frac{1}{3} - \frac{\sqrt{2}}{12}\end{aligned}$$

所以原式等於

$$-\frac{1}{3}\left(-\frac{7\sqrt{2}}{12} + \ln(\sqrt{2}+1) - \frac{1}{3} + \frac{\sqrt{2}}{12}\right) = \frac{1}{9} + \frac{\sqrt{2}}{6} - \frac{1}{3}\ln(\sqrt{2}+1)$$

(4) $\displaystyle\iint_\Omega f[r,\theta]\, dA = \int_0^{\frac{\pi}{2}}\int_0^{\sqrt{\sin 2\theta}} 1\cdot r\; dr\; d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\sin 2\theta\; d\theta = -\frac{1}{4}\cos 2\theta\Big|_0^{\frac{\pi}{2}} = \frac{1}{2}$

$\boxed{\text{習題解答}}$ **6.3.6.**

(1) $r=\sin 2\theta$, 取 $0\le\theta\frac{\pi}{2}$ 之一葉. 面積等於

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_0^{\sin 2\theta} 1\,dr\,d\theta &= \int_0^{\frac{\pi}{2}}\frac{r^2}{2}\Big|_0^{\sin 2\theta}\,d\theta=\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin^2 2\theta\,d\theta\\ &= \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos 4\theta}{2}\,d\theta=\frac{1}{4}\big(\theta-\frac{\sin 4\theta}{4}\big)\big|_0^{\frac{\pi}{2}}=\frac{1}{4}\cdot\frac{\pi}{2}=\frac{\pi}{8}\end{aligned}$$

(2) 積分區域為 $\frac{\pi}{2}\theta\le\frac{3\pi}{2}$, $1\le r\le 1-\cos\theta$. 則面積為

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_1^{1-\cos\theta} 1\cdot r\,dr\,d\theta &= \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\frac{r^2}{2}\big|_1^{1-\cos\theta}\,d\theta=\frac{1}{2}\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\cos^2\theta-\cos\theta\,d\theta\\ &= \frac{1}{2}\Big(-2\sin\theta+\frac{\theta}{2}+\frac{\sin 4\theta}{4}\Big)\Big|_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\\ &= \frac{1}{2}\big((2+2)+\frac{\pi}{2}+0\big)=2+\frac{\pi}{4}\end{aligned}$$

(3) 依題意得

$$\begin{aligned}\int_{\frac{\pi}{2}}^{\pi}\int_1^2 e^{r^2}\cdot r\,dr\,d\theta &= \int_{\frac{\pi}{2}}^{\pi}d\theta\cdot\int_1^2 re^{r^2}\,dr\\ &= \frac{\pi}{2}\cdot\frac{e^{r^2}}{2}\big|_1^2=\frac{\pi}{2}\cdot\frac{e^4-e}{2}=\frac{e^4-e}{4}\pi\end{aligned}$$

(4) 依題意得

$$\begin{aligned}\int_0^{\pi}\int_0^1\ln(r^2+1)\cdot r\,dr\,d\theta &= \int_0^{\pi}d\theta\cdot\int_0^1 r\ln(r^2+1)\,dr\\ &\overset{u=r^2+1}{=} \pi\cdot\int_1^2\frac{\ln u}{2}\,du=\pi\cdot\frac{1}{2}\cdot(u\ln u-u)\big|_1^2\\ &= \frac{\pi}{2}\cdot(2\ln 2-2+1)=\frac{2\ln 2-1}{2}\pi\end{aligned}$$

(5) 積分區域相當於 $0\le\theta\le\frac{\pi}{2}$, $0\le r\cos\theta$. 所以第一項積分等於

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} r\sin\theta\cdot r\,d\theta &= \int_0^{\frac{\pi}{2}}\sin\theta\cdot\frac{r^3}{3}\Big|_0^{\cos\theta}\,d\theta=\frac{1}{3}\int_0^{\frac{\pi}{2}}\sin\theta\cos^3\theta\,d\theta\\ &= -\frac{1}{3}\Big(\frac{\cos^4\theta}{4}\Big)\Big|_0^{\frac{\pi}{2}}=-\frac{1}{3}\cdot\big(0-\frac{1}{4}\big)=\frac{1}{12}\end{aligned}$$

另一積分為

$$\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} r\cos\theta\cdot r\,d\theta=\int_0^{\frac{\pi}{2}}\cos\theta\cdot\frac{r^3}{3}\Big|_0^{\cos\theta}\,d\theta=\frac{1}{3}\int_0^{\frac{\pi}{2}}\cos^4\theta\,d\theta$$

但

$$\begin{aligned}\int\cos^4\theta\,d\theta &= \int\Big(\frac{1+\cos 2\theta}{2}\Big)^2\,d\theta=\frac{1}{4}\int 1+2\cos 2\theta+\frac{1+\cos 4\theta}{2}\,d\theta\\ &= \frac{1}{4}\int\frac{3}{2}+2\cos 2\theta+\frac{\cos 4\theta}{2}\,d\theta\\ &= \frac{3}{8}\theta+\sin 2\theta+\frac{\sin 4\theta}{8}+C\end{aligned}$$

原式等於

$$\frac{1}{3}\cdot\Big(\frac{3}{8}\theta+\sin 2\theta+\frac{\sin 4\theta}{8}\Big)\Big|_0^{\frac{\pi}{2}}=\frac{1}{3}\cdot\frac{3}{8}\cdot\frac{\pi}{2}=\frac{\pi}{16}$$

習題解答 6.3.7.

(1) $x = 2$ 相當於 $r\cos\theta = 2 \Rightarrow r = \dfrac{2}{\cos\theta}$, $y = x$ 則是 $\theta = \dfrac{\pi}{4}$, 所以區域 $0 \le x \le 2$, $0 \le y \le x$, 極坐標表示為 $0 \le \theta \le \dfrac{\pi}{4}$, $0 \le r \le \dfrac{2}{\cos\theta}$.

(2)

$$\begin{aligned}
&\int_0^{\frac{\pi}{4}}\int_0^{\frac{2}{\cos\theta}} r\sin\theta \cdot r\,dr\,d\theta = \int_0^{\frac{\pi}{4}} \sin\theta \cdot \Big(\frac{r^3}{3}\Big)\Big|_0^{\frac{2}{\cos\theta}}\,d\theta \\
= \;& \frac{8}{3}\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^3\theta}\,d\theta = \frac{8}{3}\cdot(-1)\cdot(-\frac{1}{2})\cdot\frac{1}{\cos^2\theta}\Big|_0^{\frac{\pi}{4}} \\
= \;& \frac{4}{3}\cdot(2-1) = \frac{4}{3}
\end{aligned}$$

習題解答 6.3.9.

易知下列區域有底下的包含關係: $x^2 + y^2 \le 1 \subset \Omega \subset x^2 + y^2 \le 2$, 又 $\dfrac{1}{1+x^2+y^2} > 0$, 所以

$$\iint_{x^2+y^2\le 1} \frac{1}{1+x^2+y^2}\,dA < \iint_\Omega \frac{1}{1+x^2+y^2}\,dA < \iint_{x^2+y^2\le 2} \frac{1}{1+x^2+y^2}\,dA$$

但

$$\begin{aligned}
\iint_{x^2+y^2\le 1} \frac{1}{1+x^2+y^2}\,dA &= \int_0^{2\pi}\int_0^1 \frac{1}{1+r^2}\cdot r\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^1 \frac{r}{1+r^2}\,dr = 2\pi\cdot\frac{\ln(1+r^2)}{2}\Big|_0^1 = \pi\ln 2
\end{aligned}$$

$$\begin{aligned}
\iint_{x^2+y^2\le 2} \frac{1}{1+x^2+y^2}\,dA &= \int_0^{2\pi}\int_0^{\sqrt{2}} \frac{1}{1+r^2}\cdot r\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \cdot \int_0^{\sqrt{2}} \frac{r}{1+r^2}\,dr = 2\pi\cdot\frac{\ln(1+r^2)}{2}\Big|_0^{\sqrt{2}} = \pi\ln 3
\end{aligned}$$

所以

$$\pi\ln 2 < \iint_\Omega \frac{1}{1+x^2+y^2}\,dA < \pi\ln 3$$