6.2 Fubini定理

習題解答 **6.2.4.**

(1)

$$\begin{aligned}\int_{-1}^{1}\left(\int_{0}^{\frac{1}{1+x^2}} e^{y+yx^2}dy\right)dx &= \int_{-1}^{1}\frac{1}{1+x^2}\left(e^{y(1+x^2)}\Big|_0^{\frac{1}{1+x^2}}\right)dx\\ &= \int_{-1}^{1}\frac{(e-1)}{1+x^2}\,dx = (e-1)\tan^{-1}x\Big|_{-1}^{1} = \frac{\pi}{2}(e-1)\end{aligned}$$

(2)

$$\begin{aligned}\int_0^{\pi}\left(\int_0^{\sin y}\sin y\,dx\right)dy &= \int_0^{\pi}\sin y\cdot x\Big|_0^{\sin y}dy = \int_0^{\pi}\sin^2 y\,dy\\ &= \int_0^{\pi}\frac{1-\cos 2y}{2}\,dy = \left(\frac{y}{2}-\frac{\sin 2y}{4}\right)\Big|_0^{\pi} = \frac{\pi}{2}\end{aligned}$$

(3)

$$\begin{aligned}\int_0^{\frac{\pi}{2}}\left(\int_0^{x} x\cos y\,dy\right)dx &= \int_0^{\frac{\pi}{2}} x\cdot\sin y\Big|_0^{x}dx = \int_0^{\frac{\pi}{2}} x\sin x\,dx\\ &= \int_0^{\frac{\pi}{2}} xd(-\cos x) = (-x\cos x+\sin x)\Big|_0^{\frac{\pi}{2}}\\ &= 1\end{aligned}$$

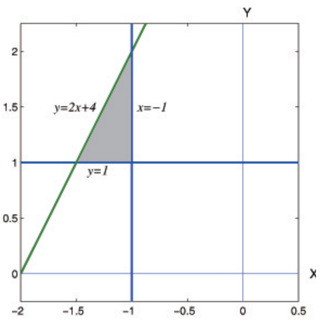
(4)

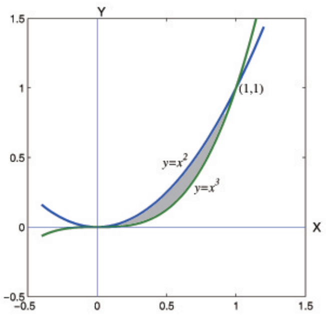
$$\begin{aligned}\int_0^1\int_0^{\sqrt{1-x^2}} xdy\,dx &= \int_0^1 x\left(y|_0^{\sqrt{1-x^2}}\right)dx\\ &= \int_0^1 x\sqrt{1-x^2}\,dx = \frac{-1}{3}(1-x^2)^{\frac{3}{2}}\Big|_0^1\\ &= \frac{1}{3}\end{aligned}$$

習題解答 **6.2.5.**

(1) 將區域畫出如下左圖, $y=1$ 與 $\frac{y}{2}-2=x$ 交於 $x=-\frac{3}{2}$ 處. 所以此區域相當於

$$-\frac{3}{2}\le x\le -1,\ \ 1\le y\le 2x+4$$



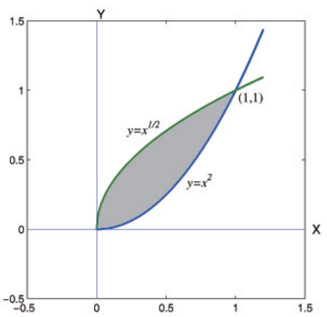


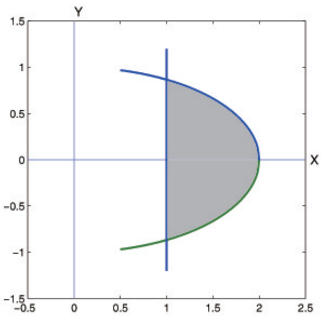
(2) 將區域畫出如上右圖, 兩曲線交於 $(0,0)$ 和 $(1,1)$ 處. 所以此區域相當於

$$0\le y\le 1,\ \ y^{\frac{1}{2}}\le x\le y^{\frac{1}{3}}$$

(3) 將區域畫出如下頁左圖, 兩曲線交於 $(0,0)$ 和 $(1,1)$ 處. 所以此區域相當於

$$0\le y\le 1,\ \ y^{2}\le x\le y^{\frac{1}{2}}$$





(4) 將區域畫出如上右圖, $y=\frac{\sqrt{3}}{2}$ 與 $x=2\sqrt{1-y^2}$ 交於 $(1,\frac{\sqrt{3}}{2})$；$y=\frac{-\sqrt{3}}{2}$ 與 $x=2\sqrt{1-y^2}$ 交於 $(1,-\frac{\sqrt{3}}{2})$. 又 $x=2\sqrt{1-y^2}$ 與 x-軸交於 $(2,0)$. 所以此區域相當於

$$1\le x\le 2,\ \ -\sqrt{1-\frac{x^2}{4}}\le y\le\sqrt{1-\frac{x^2}{4}}$$

$\boxed{\text{習題解答}}$ **6.2.6.**

(1)

$$\begin{aligned}\int_0^1\int_y^1 x^2e^{xy}\,dx\,dy &= \int_0^1\int_0^x x^2e^{xy}\,dy\,dx=\int_0^1 x\left(e^{xy}|_0^x\right)dx\\ &= \int_0^1 xe^{x^2}-x\,dx=\frac{1}{2}\left(e^{x^2}-x^2\right)\Big|_0^1=\frac{1}{2}e-1\end{aligned}$$

(2)

$$\begin{aligned}\int_0^1\int_{x^{\frac{1}{3}}}^1 \frac{1}{1+y^4}\,dy\,dx &= \int_0^1\int_0^{y^3}\frac{1}{1+y^4}\,dx\,dy=\int_0^1\frac{1}{1+y^4}\left(x|_0^{y^3}\right)dy\\ &= \int_0^1\frac{y^3}{1+y^4}dy=\frac{1}{4}\ln|1+y^4|\Big|_0^1=\frac{1}{4}\ln 2\end{aligned}$$

(3)

$$\begin{aligned}\int_0^2\int_x^2 y^2\sin xy\,dy\,dx &= \int_0^2\int_0^y y^2\sin xy\,dx\,dy=\int_0^2 y\left(-\cos xy|_0^y\right)dy\\ &= \int_0^2 y-y\cos(y^2)dy=\frac{1}{2}y^2-\frac{1}{2}\sin(y^2)\Big|_0^2=2-\frac{1}{2}\sin 4\end{aligned}$$

(4)

$$\begin{aligned}\int_0^1\int_{\sqrt{y}}^1 \frac{1}{x(1+x^2)}\,dx\,dy &= \int_0^1\int_0^{x^2}\frac{1}{x(1+x^2)}\,dy\,dx=\int_0^1\frac{1}{x(1+x^2)}\left(y|_0^{x^2}\right)dx\\ &= \int_0^1\frac{x}{1+x^2}\,dx=\frac{1}{2}\ln|1+x^2|\Big|_0^1=\frac{1}{2}\ln 2\end{aligned}$$

$\boxed{\text{習題解答}}$ **6.2.7.**

(1) 用 x-軸將此區域割成兩區域, 並用 y 方向的積分形式.

$$\begin{aligned}
&\int_{-1}^{0}\int_{-y}^{\sqrt{y+2}} xy\,dx\,dy+\int_{0}^{2}\int_{y}^{\sqrt{y+2}} xy\,dx\,dy\\
=\ &\int_{-1}^{0} y\Big(\frac{x^2}{2}\Big|_{-y}^{\sqrt{y+2}}\Big)\,dy+\int_{-1}^{0} y\Big(\frac{x^2}{2}\Big|_{y}^{\sqrt{y+2}}\Big)\,dy\\
=\ &\frac{1}{2}\int_{-1}^{0} y^2+2y-y^3\,dy+\frac{1}{2}\int_{0}^{2} y^2+2y-y^3\,dy\\
=\ &\frac{1}{2}\Big(-\frac{y^4}{4}+\frac{y^3}{3}+y^2\Big|_{-1}^{2}\Big)\\
=\ &\frac{1}{2}\Big(-\frac{16}{4}+\frac{8}{3}+4-\big(-\frac{1}{4}-\frac{1}{3}+1\big)\Big)=\frac{1}{2}\cdot\frac{9}{4}=\frac{9}{8}
\end{aligned}$$

(2) 分兩種想法說明如下

第一種作法: 此區域對 x-軸與 y-軸皆對稱, 又

$$\iint_\Omega y-2x\,dA=\iint_\Omega y\,dA-2\iint_\Omega x\,dA$$

前者因對 x-軸對稱, 而 y 為奇函數, 所以積分為 0, 同理, 後者對 y-軸對稱, 而 x 為奇函數, 所以積分也為 0, 故總積分為 0.

第二種作法: 四線各自交於 $(\pm1,0)$, $(0,\pm1)$ 四點. 以 y-軸將此區域分成兩部分.

$$\begin{aligned}
&\int_{-1}^{0}\int_{-x-1}^{x+1} y-2x\,dy\,dx+\int_{0}^{1}\int_{x-1}^{-x+1} y-2x\,dy\,dx\\
=\ &\int_{-1}^{0}\big(\frac{y^2}{2}-2xy\big)\Big|_{-x-1}^{x+1}\,dx+\int_{0}^{1}\big(\frac{y^2}{2}-2xy\big)\Big|_{x-1}^{-x+1}\,dx\\
=\ &\int_{-1}^{0}-4x^2-4x\,dx+\int_{0}^{1}4x^2-4x\,dx\\
=\ &\big(\frac{-4x^3}{3}-2x^2\big)\Big|_{-1}^{0}+\big(\frac{4x^3}{3}-2x^2\big)\Big|_{0}^{1}\\
=\ &-\frac{4}{3}+2+\frac{4}{3}-2=0
\end{aligned}$$

(3) 三線交如右圖. 因為 x 方向必須分成兩區域, 因此選 y 方向做積分.

$$\begin{aligned}
&\int_{0}^{1}\int_{y-1}^{1-y} x+\sqrt{y}\,dx\,dy=\int_{0}^{1}\big(\frac{x^2}{2}+\sqrt{y}x\big)\Big|_{y-1}^{1-y}\,dy\\
=\ &\int_{0}^{1} 0+2(y^{\frac{1}{2}})-y^{\frac{3}{2}}\,dy=2\big(\frac{2}{3}y^{\frac{3}{2}}-\frac{2}{5}y^{\frac{5}{2}}\big)\Big|_{0}^{1}\\
=\ &2\big(\frac{2}{3}-\frac{2}{5}-0\big)=\frac{8}{15}
\end{aligned}$$