5.6 極值測試與應用

習題解答 5.6.2.

(4) $f(x,y)=xe^{-(x^2+y^2)}$. $\nabla f(x,y)=((1-2x^2)e^{-(x^2+y^2)},-2xye^{-(x^2+y^2)})=(0,0)$. 由

$$\begin{cases} 1-2x^2 &= 0 \\ -2xy &= 0 \end{cases} \Rightarrow x=\pm\frac{1}{\sqrt{2}},\ y=0$$

所以 $(\pm\frac{1}{\sqrt{2}},0)$ 為候選點. 但

$$\begin{aligned} D(x,y) &= (e^{-(x^2+y^2)})^2\cdot\begin{vmatrix} (-4x-2x(1-2x^2)) & -2y(1-2x^2) \\ -2y(1-2x^2) & -2x(1-2y^2)\end{vmatrix} \\ \Rightarrow \quad D(\tfrac{1}{\sqrt{2}},0) &= e^{-1}\cdot\begin{vmatrix} \frac{-4}{\sqrt{2}} & 0 \\ 0 & \frac{-2}{\sqrt{2}}\end{vmatrix}=4e^{-1} \\ D(-\tfrac{1}{\sqrt{2}},0) &= e^{-1}\cdot\begin{vmatrix} \frac{4}{\sqrt{2}} & 0 \\ 0 & \frac{2}{\sqrt{2}}\end{vmatrix}=4e^{-1}\end{aligned}$$

1. $D(\frac{1}{\sqrt{2}},0)=4e^{-1}>0$, $\frac{-4}{\sqrt{2}}<0$, 所以 $f(\frac{1}{\sqrt{2}},0)=\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$ 是極大值.
2. $D(-\frac{1}{\sqrt{2}},0)=4e^{-1}>0$, $\frac{4}{\sqrt{2}}>0$, 所以 $f(-\frac{1}{\sqrt{2}},0)=-\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$ 是極小值.

習題解答 5.6.3.

(4) $f(x,y)=x^2+\lambda xy+y^2$. $\nabla f(x,y)=(2x+\lambda y,\lambda x+2y)=(0,0)$ $\Rightarrow$ 除了 $\lambda=\pm 2$ 之外, 候選點只有 $(0,0)$. 此時

$$D(x,y)=\begin{vmatrix} 2 & \lambda \\ \lambda & 2\end{vmatrix}=4-\lambda^2<0$$

由極值測試, 當 $|\lambda|>2$ 時, $D(0,0)<0$, $(0,0,0)$ 為鞍點；當 $|\lambda|<2$ 時, $D(0,0)>0$, 又因為 $2>0$, 所以 $f(0,0)=0$ 為極小值.

當 $\lambda=\pm 2$ 時, $f(x,y)=(x\pm y)^2\geq 0$, $f(x,y)$ 在 $x\pm y=0$ 時, 有極小值 0.

習題解答 **5.6.9.**

用最小平方法, 將誤差寫為

$$E(\alpha,\beta)=\sum_{i=1}^{n}(\alpha a_i^2+\beta a_i+10-b_i)^2$$

求候選點如下

$$\begin{aligned}\frac{\partial E}{\partial \alpha} &= \sum_{i=1}^{n}2(\alpha a_i^2+\beta a_i+10-b_i)\cdot a_i^2\\ &= 2\sum_{i=1}^{n}(\alpha a_i^4+\beta a_i^3+10a_i^2-a_i^2 b_i)=0\\ \frac{\partial E}{\partial \beta} &= \sum_{i=1}^{n}2(\alpha a_i^2+\beta a_i+10-b_i)\cdot a_i\\ &= 2\sum_{i=1}^{n}(\alpha a_i^3+\beta a_i^2+10a_i-a_i b_i)=0\end{aligned}$$

用平均記號, 則原式相當於

$$\begin{cases}\overline{a^4}+\overline{a^3} = \overline{a^2 b}-10\,\overline{a^2}\\ \overline{a^3}+\overline{a^2} = \overline{ab}-10\,\overline{a}\end{cases}$$

計算資料如下

	a	b	a^2	ab	a^3	a^2b	a^4
	1	1	1	1	1	1	1
	1	3	1	3	1	3	1
	2	0	4	0	8	0	16
	2	1	4	2	8	4	16
	3	3	9	9	27	27	81
+)	4	10	16	40	64	160	256
	13	18	35	55	109	195	371
平均	$\frac{13}{6}$	$\frac{18}{6}$	$\frac{35}{6}$	$\frac{55}{6}$	$\frac{109}{6}$	$\frac{195}{6}$	$\frac{371}{6}$

代入方程乘以 6 後得

$$371\alpha+109\beta=195-350=-155$$
$$109\alpha+35\beta=55-130=-75$$

解得

$$\alpha=\frac{2750}{1104}\approx 2.49,\quad \beta=\frac{-10930}{1104}\approx -9.9$$

因此最佳的二次曲線為 $y=2.49x^2-9.9x+10$.

習題解答 **5.6.11.**

$P(t) = B - At$, $A, B > 0$, 且 $0 \le t \le \frac{A}{B} \equiv \alpha$. 現依課本方法分析：

(1)（不合作）定義

$$\begin{aligned} R_X(x,y) &= (A - B(x+y))x = Ax - Bx^2 - Bxy \\ R_Y(x,y) &= (A - B(x+y))y = Ay - Bxy - By^2 \\ \frac{\partial R_x}{\partial x} &= A - 2Bx - By = 0 \\ \frac{\partial R_Y}{\partial y} &= A - Bx - 2by = 0 \end{aligned}$$

解得 $(x,y) = (\frac{\alpha}{3}, \frac{\alpha}{3})$, 且獲益為 $R_X(\frac{\alpha}{3}, \frac{\alpha}{3}) = \frac{\alpha^2}{9}B$.

(2)（合作）依前習題, 可將問題化成單變數, 令

$$R(t) = \frac{P(2t)\cdot 2t}{2} = At - 2Bt^2 \ \Rightarrow\ R'(t) = A - 4Bt = 0 \ \Rightarrow\ t = \frac{\alpha}{4}$$

此時獲益為 $R(\frac{\alpha}{4}) = \frac{\alpha^2}{8}B$

(3)（背叛）假設 Y 背叛, 因此 X 仍生產 $\frac{\alpha}{4}$ 單位, Y 則試圖最大化其獲益函數

$$\begin{aligned} R_Y(y) &= (A - B(\frac{\alpha}{4} + y))y = B(\frac{3\alpha}{4}y - y^2) \\ \Rightarrow\ R_Y'(y) &= B(\frac{3\alpha}{4} - 2y) = 0 \ \Rightarrow\ y = \frac{3\alpha}{8} \end{aligned}$$

由此得 Y 穫益為 $B\cdot(\frac{3\alpha}{4} - \frac{3\alpha}{8})\cdot\frac{3\alpha}{8} = \frac{9\alpha^2}{64}B$,

而 X 的穫益為 $B\cdot(\frac{3\alpha}{4} - \frac{3\alpha}{8})\cdot\frac{\alpha}{4} = \frac{3\alpha^2}{32}B$.

現比較各種情況穫益, 發現

$$\frac{9}{64} > \frac{1}{8} > \frac{1}{9} > \frac{3}{32}$$

且

$$\frac{9}{64} + \frac{3}{32} < \frac{1}{8}\cdot 2$$

滿足課本對囚犯悖論所要求的兩個條件.