5.5 高階偏導數與泰勒展式

$\boxed{\text{習題解答}}$ **5.5.1.**

(4)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \ln y y^x, \quad \frac{\partial f}{\partial y} = xy^{x-1} \\
\frac{\partial^2 f}{\partial x^2} &= (\ln x)^2 y^x \\
\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} &= \frac{1}{y}\cdot y^x + \ln y \cdot xy^{x-1} = (1 + x\ln y)y^{x-1} \\
\frac{\partial^2 f}{\partial y^2} &= x(x-1)y^{x-2}
\end{aligned}$$

$\boxed{\text{習題解答}}$ **5.5.3.** $z = f(x, y)$, $\vec{u} = (\cos\theta, \sin\theta)$, 說明

$$\frac{\partial^2 f}{\partial \vec{u}^2} \equiv \frac{\partial(\frac{\partial f}{\partial \vec{u}})}{\partial \vec{u}} = \frac{\partial^2 f}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 f}{\partial x\,\partial y}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta$$

先定義

$$\frac{\partial f}{\partial \vec{u}}(x, y) = \nabla f(x, y)\cdot\vec{u} = \frac{\partial f}{\partial x}(x, y)\cos\theta + \frac{\partial f}{\partial y}(x, y)\sin\theta$$

(也就是對任何點, 都做同一個向量 $\vec{u}$ 的方向導數所得的函數.) 所以

$$\nabla(\frac{\partial f}{\partial \vec{u}}) = \Big(\frac{\partial^2 f}{\partial x^2}\cos\theta + \frac{\partial^2 f}{\partial x\,\partial y}\sin\theta,\ \ \frac{\partial^2 f}{\partial x\,\partial y}\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin\theta\Big)$$

因此

$$\begin{aligned}
&\frac{\partial^2 f}{\partial \vec{u}^2} \equiv \frac{\partial(\frac{\partial f}{\partial \vec{u}})}{\partial \vec{u}} = \nabla(\frac{\partial f}{\partial \vec{u}})\cdot\vec{u} \\
=\ & \Big(\frac{\partial^2 f}{\partial x^2}\cos\theta + \frac{\partial^2 f}{\partial x\,\partial y}\sin\theta,\ \ \frac{\partial^2 f}{\partial x\,\partial y}\cos\theta + \frac{\partial^2 f}{\partial y^2}\sin\theta\Big)\cdot(\cos\theta, \sin\theta) \\
=\ & \frac{\partial^2 f}{\partial x^2}\cos^2\theta + 2\frac{\partial^2 f}{\partial x\,\partial y}\cos\theta\sin\theta + \frac{\partial^2 f}{\partial y^2}\sin^2\theta
\end{aligned}$$

$\boxed{\text{習題解答}}$ **5.5.4.**

(5) 是諧和函數.

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{\partial\big(\frac{x}{x^2+y^2}\big)}{\partial x} = \frac{x^2+y^2-2x^2}{x^2+y^2} = \frac{y^2-x^2}{x^2+y^2} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{\partial\big(\frac{y}{x^2+y^2}\big)}{\partial y} = \frac{x^2-y^2}{x^2+y^2} \\
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} &= \frac{(y^2-x^2)+(x^2-y^2)}{x^2+y^2} = 0
\end{aligned}$$

$\boxed{\text{習題解答}}$ **5.5.8.**

(1)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= 3x^2+2xy-3y^2 \;\Rightarrow\; \frac{\partial f}{\partial x}(1,2)=-5\\
\frac{\partial f}{\partial y} &= x^2-6xy+3y^2 \;\Rightarrow\; \frac{\partial f}{\partial y}(1,2)=1\\
\frac{\partial^2 f}{\partial x^2} &= 6x+2y \;\Rightarrow\; \frac{\partial^2 f}{\partial x^2}(1,2)=10\\
\frac{\partial^2 f}{\partial x\,\partial y} &= 2x-6y \;\Rightarrow\; \frac{\partial^2 f}{\partial x\,\partial y}(1,2)=-10\\
\frac{\partial^2 f}{\partial y^2} &= -6x+6y \;\Rightarrow\; \frac{\partial^2 f}{\partial y^2}(1,2)=6
\end{aligned}$$

又

$$\frac{\partial f}{\partial x^3}=6,\;\; \frac{\partial f}{\partial x^2\partial y}=6,\;\; \frac{\partial f}{\partial x\partial y^2}=-6,\;\; \frac{\partial f}{\partial y^3}=6,$$

且 $f(1,2)=-1$. 故其三階泰勒多項式為

$$\begin{aligned}
P_3(x,y) &= -1-5(x-1)+(y-2)+\frac{1}{2}\big(10(x-1)^2\\
&\quad +2\cdot(-10)(x-1)(y-2)+6(y-2)^2\big)\\
&\quad +\frac{1}{6}\big(6(x-1)^3+3\cdot 2(x-1)^2(y-2)\\
&\quad +3\cdot(-6)(x-1)(y-2)^2+6(y-2)^3\big)\\
&= -1-5(x-1)+(y-2)\\
&\quad +5(x-1)^2-10(x-1)(y-2)+3(y-2)^2\\
&\quad +(x-1)^3+(x-1)^2(y-2)-3(x-1)(y-2)^2+(y-2)^3
\end{aligned}$$

$\boxed{\text{習題解答}}$ **5.5.9.**

(3) 猜測:

$$\sin x\cos y \sim \Big(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots\Big)\Big(1-\frac{y^2}{2!}+\frac{y^4}{4!}+\cdots\Big)=x-\frac{xy^2}{2}+\frac{x^3}{6}$$

$f(0,0)=0$, 且

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \cos x\cos y \;\Rightarrow\; \frac{\partial f}{\partial x}(0,0)=1\\
\frac{\partial f}{\partial y} &= -\sin x\sin y \;\Rightarrow\; \frac{\partial f}{\partial y}(0,0)=0\\
\frac{\partial^2 f}{\partial x^2} &= -\sin x\cos y \;\Rightarrow\; \frac{\partial^2 f}{\partial x^2}(0,0)=0\\
\frac{\partial^2 f}{\partial x\,\partial y} &= -\cos x\sin y \;\Rightarrow\; \frac{\partial^2 f}{\partial x\,\partial y}(0,0)=0\\
\frac{\partial^2 f}{\partial y^2} &= -\sin x\cos y \;\Rightarrow\; \frac{\partial^2 f}{\partial y^2}(0,0)=0
\end{aligned}$$

所以其二階泰勒多項式為符合猜測的

$$P_2(x,y)=0+x+0=x$$

(6) 猜測: $\sqrt{1+x^2+y^2} \sim 1+\frac{1}{2}(x^2+y^2)+\mathbf{C}_2^{\frac{1}{2}}(x^2+y^2)^2+\cdots$. $f(0,0)=1$, 且

$$\begin{aligned}
\frac{\partial f}{\partial x} &= x(1+x^2+y^2)^{-\frac{1}{2}} \Rightarrow \frac{\partial f}{\partial x}(0,0)=0 \\
\frac{\partial f}{\partial y} &= y(1+x^2+y^2)^{-\frac{1}{2}} \Rightarrow \frac{\partial f}{\partial y}(0,0)=0 \\
\frac{\partial^2 f}{\partial x^2} &= (1-x^2)(1+x^2+y^2)^{-\frac{3}{2}} \Rightarrow \frac{\partial^2 f}{\partial x^2}(0,0)=1 \\
\frac{\partial^2 f}{\partial x\,\partial y} &= -xy(1+x^2+y^2)^{-\frac{3}{2}} \Rightarrow \frac{\partial^2 f}{\partial x\,\partial y}(0,0)=0 \\
\frac{\partial^2 f}{\partial y^2} &= (1-y^2)(1+x^2+y^2)^{-\frac{3}{2}} \Rightarrow \frac{\partial^2 f}{\partial y^2}(0,0)=1
\end{aligned}$$

所以其二階泰勒多項式為符合猜測的

$$P_2(x,y)=1+0+\frac{1}{2}(x^2+2\cdot 0xy+y^2)=1+\frac{1}{2}(x^2+y^2)$$