5.4 方向導數與梯度

習題解答 **5.4.3.**

由於梯度和等高線 $f(x,y)=C$ 垂直, 考慮等高線上一點 (x_0,y_0), 則

$$\nabla f(x_0,y_0)=(\frac{\partial f}{\partial x}(x_0,y_0),\frac{\partial f}{\partial y}(x_0,y_0))$$

為過 (x_0,y_0) 切線之法向量, 因此切線方程式為

$$\frac{\partial f}{\partial x}(x_0,y_0)(x-x_0)+\frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)=0$$

習題解答 **5.4.7.**

(1) $\nabla f=(ye^{xy},xe^{xy}),\quad \vec{u}=(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),$

$$\frac{\partial f}{\partial \vec{u}}=\nabla f(0,0)\cdot\vec{u}=(0,0)\cdot(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=0$$

(3) $\nabla f=(\frac{y}{y^2+x^2},\frac{-x}{y^2+x^2}),\quad \vec{u}=(\frac{1}{2},\frac{\sqrt{3}}{2}),$

$$\frac{\partial f}{\partial \vec{u}}=\nabla f(1,1)\cdot\vec{u}=(\frac{1}{2},\frac{-1}{2})\cdot(\frac{1}{2},\frac{\sqrt{3}}{2})=\frac{1-\sqrt{3}}{4}$$

(4) $\nabla f=(\frac{2x}{x^2+y^4},\frac{4y^3}{x^2+y^4}),\quad \vec{u}=(\frac{-2}{\sqrt{5}},\frac{1}{\sqrt{5}}),$

$$\frac{\partial f}{\partial \vec{u}}=\nabla f(1,0)\cdot\vec{u}=(2,0)\cdot(\frac{-2}{\sqrt{5}},\frac{1}{\sqrt{5}})=-\frac{4}{\sqrt{5}}$$

(6) $\nabla f=(z\cos xz,\frac{1}{z},\frac{-y}{z^2}+x\cos xz),\quad \vec{u}=(\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}},\frac{3}{\sqrt{14}}),$

$$\frac{\partial f}{\partial \vec{u}}=\nabla f(\pi,2,2)\cdot\vec{u}=(2,\frac{1}{2},\frac{-1}{2}+\pi)\cdot(\frac{2}{\sqrt{14}},\frac{1}{\sqrt{14}},\frac{3}{\sqrt{14}})=\frac{3+3\pi}{\sqrt{14}}$$

習題解答 **5.4.8.**

$(1,1)$ 方向的單位向量為 $(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$; $(-1,1)$ 方向的單位向量為 $(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})$. 若令 $A=\frac{\partial f}{\partial x}(a,b)$, $B=\frac{\partial f}{\partial y}(a,b)$, 由題意知

$$\begin{cases}(A,B)\cdot(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=\frac{A}{\sqrt{2}}+\frac{B}{\sqrt{2}}=1\\(A,B)\cdot(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}})=\frac{-A}{\sqrt{2}}+\frac{B}{\sqrt{2}}=2\end{cases}$$

解得 $A=-\frac{\sqrt{2}}{2}$, $B=\frac{3\sqrt{2}}{2}$, 故 $\nabla f(a,b)=(-\frac{\sqrt{2}}{2},\frac{3\sqrt{2}}{2})$

習題解答 **5.4.9.**

(3)

$$\begin{aligned}\frac{\partial f}{\partial x}&=y\cos xy &\Rightarrow\quad& \frac{\partial f}{\partial x}(\pi,\frac{1}{6},\pi)=\frac{\sqrt{3}}{12}\\\frac{\partial f}{\partial y}&=x\cos xy+z\cos yz &\Rightarrow\quad& \frac{\partial f}{\partial y}(\pi,\frac{1}{6},\pi)=\sqrt{3}\pi\\\frac{\partial f}{\partial z}&=y\cos yz &\Rightarrow\quad& \frac{\partial f}{\partial z}(\pi,\frac{1}{6},\pi)=\frac{\sqrt{3}}{12}\end{aligned}$$

方程式為 $\frac{\sqrt{3}}{12}(x-\pi)+\sqrt{3}\pi(y-\frac{1}{6})+\frac{\sqrt{3}}{12}(z-\pi)=0$, 即 $x+12\pi y+z=4\pi$.

(4)

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{yz}{1+x^2y^2z^2} \quad \Rightarrow \quad \frac{\partial f}{\partial x}(2,\frac{1}{3},\frac{3}{2}) = \frac{1}{4} \\
\frac{\partial f}{\partial y} &= \frac{xz}{1+x^2y^2z^2} \quad \Rightarrow \quad \frac{\partial f}{\partial y}(2,\frac{1}{3},\frac{3}{2}) = \frac{3}{2} \\
\frac{\partial f}{\partial z} &= \frac{xy}{1+x^2y^2z^2} \quad \Rightarrow \quad \frac{\partial f}{\partial z}(2,\frac{1}{3},\frac{3}{2}) = \frac{1}{3}
\end{aligned}$$

方程式為 $\frac{1}{4}(x-2)+\frac{3}{2}(y-\frac{1}{3})+\frac{1}{3}(z-\frac{3}{2})=0$, 即 $3x+18y+4z=18$.