4.4 泰勒定理的應用

習題解答 **4.4.2.**

(1) 計算如下

$$\begin{aligned} f'(x) &= -\sin x + x = 0 \quad \Rightarrow \quad x = 0 \qquad \text{(第二章第二節例題)} \\ f''(x) &= -\cos x + 1 \quad \Rightarrow \quad f''(0) = 0 \\ f'''(x) &= \sin x \quad \Rightarrow \quad f'''(0) = 0 \\ f^{(4)}(x) &= \cos x \quad \Rightarrow \quad f^{(4)}(0) = 1 > 0 \end{aligned}$$

所以連續函數 $f(x) = \cos x - (1 - \frac{x^2}{2})$ 在 $x = 0$ 有唯一極小值, 因此是最小值.

(2) 由於 $f(x) = \cos x - (1 - \frac{x^2}{2})$ 在 $x = 0$ 為最小值, 因此 $f(x) \geq f(0) = 0$.

習題解答 **4.4.4.**

(2) $\displaystyle\lim_{x\to 0} \frac{x - \tan^{-1}x}{x^3} = \lim_{x\to 0} \frac{1 - \frac{1}{1+x^2}}{3x^2} = \lim_{x\to 0} \frac{1}{3(1+x^2)} = \frac{1}{3}$

(5) $\displaystyle\lim_{x\to 0} \frac{7^x - 1}{2^x - 1} = \lim_{x\to 0} \frac{\ln 7 \cdot 7^x}{\ln 2 \cdot 2^x} = \frac{\ln 7}{\ln 2}$

(7) $\displaystyle\lim_{x\to 0} \frac{x^2}{\ln \sec x} = \lim_{x\to 0} \frac{2x}{\frac{\sec x \tan x}{\sec x}} = 2\lim_{x\to 0} \frac{x}{\tan x} = 2\lim_{x\to 0} \frac{1}{\sec^2 x} = 2$

(8) $\displaystyle\lim_{x\to 0} \frac{e^{\sin x} - 1}{\sin x} = \lim_{x\to 0} \frac{\cos x \cdot e^{\sin x}}{\cos x} = 1$

習題解答 **4.4.5.**

(1) $\displaystyle\lim_{x\to\infty} \frac{\log_3 x}{\ln(x^2+1)} = \lim_{x\to\infty} \frac{\frac{1}{\ln 3 \cdot x}}{\frac{2x}{x^2+1}} = \frac{1}{2\ln 3} \lim_{x\to\infty} \frac{x^2+1}{x^2} = \frac{1}{2\ln 3}$

(2) $\displaystyle\lim_{x\to\infty} \frac{\int_1^x \ln x \, dx}{x \ln x} = \lim_{x\to\infty} \frac{\ln x}{\ln x + x \cdot \frac{1}{x}} = 1$

(3) $\displaystyle\lim_{x\to\infty} \frac{x^2 + x + 1}{e^x} = \lim_{x\to\infty} \frac{2x+1}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = 0$

(4) $\displaystyle\lim_{x\to\infty} \frac{\ln^3 x}{x} = \lim_{x\to\infty} \frac{3\frac{\ln^2 x}{x}}{1} = \lim_{x\to\infty} \frac{6\frac{\ln x}{x}}{1} = 6\lim_{x\to\infty} \frac{\frac{1}{x}}{1} = 0$

習題解答 **4.4.6.**

(2) $\displaystyle\lim_{x\to 0} (\frac{1}{x^2} - \frac{1}{\sin x}) = \lim_{x\to 0} \frac{\sin x - x^2}{x^2 \sin x} = \lim_{x\to 0} \frac{\cos x - 2x}{2x\sin x + x^2 \cos x} = \lim_{x\to 0} \frac{1}{0}$,
故極限不存在.

(5) $\displaystyle\lim_{x\to 0} x^x = e^{\lim_{x\to 0} x \ln x}$, 但

$$\lim_{x\to 0} x \ln x = \lim_{x\to 0} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} (-x) = 0$$

所以

$$\lim_{x\to 0} x^x = e^{\lim_{x\to 0} x \ln x} = e^0 = 1$$

(6) $\displaystyle\lim_{x\to\infty} (1+x)^{\frac{1}{x}} = e^{\lim_{x\to\infty} \frac{\ln(1+x)}{x}} = e^{\lim_{x\to\infty} \frac{1}{1+x}} = e^0 = 1$