4.3 常用函數的泰勒展式

習題解答 **4.3.1.**

$$
\begin{aligned}
e &= 1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}+\cdots \\
&< 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\frac{1}{2^n}+\cdots \\
&= 1+\frac{1}{1-\frac{1}{2}}=3
\end{aligned}
$$

習題解答 **4.3.2.**

由課本知

$$
\begin{aligned}
|e-P_n(1)| &= |e-(1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!})| \\
&\leq \frac{e\cdot 1}{(n+1)!}\leq\frac{3}{(n+1)!}
\end{aligned}
$$

現希望誤差小於 $\frac{1}{10000}$, 取 n 滿足

$$
\frac{3}{(n+1)!}\leq\frac{1}{10000}\quad\Rightarrow\quad(n+1)!\geq 30000
$$

取 $n=7$ 即可.

習題解答 **4.3.3.**

$f(x)=\cos x$, 則

$$
\begin{aligned}
1 &= f(0)=-f''(0)=f^{(4)}(0)=\cdots=(-1)^n f^{(2n)}(0)=\cdots \\
0 &= f'(0)=f^{(3)}(0)=\cdots=f^{(2n+1)}(0)=\cdots
\end{aligned}
$$

由泰勒定理

$$
\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+(-1)^k\frac{x^{2k}}{(2k)!}+R_{2k+1}(x)
$$

其中

$$
|R_{2k+1}(x)|=|\frac{\sin\xi}{(2k+1)!}x^{2k+1}|\leq\frac{|x|^{2k+1}}{(2k+1)!}
$$

因此 $\lim\limits_{k\to\infty}R_{2k+1}(x)=0$, 故得

$$
\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}+\cdots+(-1)^k\frac{x^{2k}}{(2k)!}+\cdots
$$

習題解答 **4.3.6.**

(2) $x\sin x\sim x(x-\frac{x^3}{3!}+\frac{x^5}{5!}+\cdots)=x^2-\frac{x^4}{3!}+\frac{x^6}{5!}+\cdots$

(3) $e^{-x^2}\sim 1+\frac{(-x^2)}{1!}+\frac{(-x^2)^2}{2!}+\frac{(-x^2)^3}{3!}+\cdots=1-x^2+\frac{x^4}{2!}-\frac{x^6}{3!}+\cdots$

(5) 用三角恆等式

$$\begin{aligned}\sin^2 x &= \frac{1-\cos 2x}{2} = \frac{1}{2} - \frac{\cos 2x}{2}\\ &= \frac{1}{2} - \frac{1}{2}\Big(1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots\Big)\\ &= x^2 - \frac{2^3}{4!}x^4 + \frac{2^5}{6!}x^6 + \cdots\end{aligned}$$

注意：若用 $\sin x = x - \frac{x^3}{3!} + \cdots$ 自乘會很難計算.

(8)

$$\begin{aligned}\sqrt{1+x^3} = (1+x^3)^{\frac{1}{2}} &\sim 1 + \mathbf{C}_1^{\frac{1}{2}}(x^3) + \mathbf{C}_2^{\frac{1}{2}}(x^3)^2 + \mathbf{C}_3^{\frac{1}{2}}(x^3)^3 \cdots\\ &= 1 + \frac{1}{2}x^3 - \frac{1}{8}x^6 + \frac{1}{16}x^9 + \cdots\end{aligned}$$

(10) 第一種作法: 因為 $\cos^{-1} x + \sin^{-1}(x) = \dfrac{\pi}{2}$, 又已知

$$\sin^{-1} x \sim x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots$$

所以

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1}(x) = \frac{\pi}{2} - (x + \frac{x^3}{6} + \frac{3}{40}x^5 + \cdots)$$

第二種作法:

$$(\cos^{-1} x)' = \frac{-1}{\sqrt{1-x^2}} \sim -(1 - \mathbf{C}_1^{\frac{-1}{2}}x^2 + \mathbf{C}_2^{\frac{-1}{2}}x^4 - \mathbf{C}_3^{\frac{-1}{2}}x^6 + \cdots)$$

但

$$\int_0^x \frac{-1}{\sqrt{1-x^2}}\,dx = \cos^{-1} x - \cos^{-1} 0 = \cos^{-1} x - \frac{\pi}{2}$$

所以

$$\begin{aligned}\cos^{-1} x &= \frac{\pi}{2} + \int_0^x \frac{-1}{\sqrt{1-x^2}}\,dx\\ &\sim \frac{\pi}{2} + \int_0^x -(1 - \mathbf{C}_1^{\frac{-1}{2}}x^2 + \mathbf{C}_2^{\frac{-1}{2}}x^4 - \mathbf{C}_3^{\frac{-1}{2}}x^6 + \cdots)\,dx\\ &= \frac{\pi}{2} - x + \mathbf{C}_1^{\frac{-1}{2}}\frac{x^3}{3} - \mathbf{C}_2^{\frac{-1}{2}}\frac{x^5}{5} + \cdots\\ &= \frac{\pi}{2} - x - \frac{x^3}{6} - \frac{3}{40}x^5 + \cdots\end{aligned}$$

習題解答 4.3.10.

(1) 已知 $e^{\mathbf{i}\theta} = \cos\theta + \mathbf{i}\sin\theta$, $e^{-\mathbf{i}\theta} = \cos(-\theta) + \mathbf{i}\sin(-\theta) = \cos\theta - \mathbf{i}\sin\theta$, 故

$$\begin{aligned}\frac{e^{\mathbf{i}\theta} + e^{-\mathbf{i}\theta}}{2} &= \frac{(\cos\theta + \mathbf{i}\sin\theta) + (\cos\theta - \mathbf{i}\sin\theta)}{2} = \cos\theta\\ \frac{e^{\mathbf{i}\theta} - e^{-\mathbf{i}\theta}}{2\mathbf{i}} &= \frac{(\cos\theta + \mathbf{i}\sin\theta) - (\cos\theta - \mathbf{i}\sin\theta)}{2\mathbf{i}} = \sin\theta\end{aligned}$$