4.2 泰勒定理

習題解答 4.2.4.

由於 $f'(x)=g(x) \Rightarrow f^{(k+1)}(x)=g^{(k)}(x)$, 令

$$P_{n+1}(x)=\sum_{k=0}^{n+1}\frac{f^{(k)}(a)}{k!}(x-a)^k, \quad Q_n(x)=\sum_{k=0}^{n}\frac{g^{(k)}(a)}{k!}(x-a)^k$$

則

$$\begin{aligned}
P'_{n+1}(x) &= \sum_{k=1}^{n+1}\frac{k\cdot f^{(k)}(a)}{k!}(x-a)^{k-1} = \sum_{k=1}^{n+1}\frac{\cdot f^{(k)}(a)}{(k-1)!}(x-a)^{k-1}\\
&\overset{j=k-1}{=} \sum_{j=0}^{n}\frac{f^{(j+1)}(a)}{j!}(x-a)^j = \sum_{j=0}^{n}\frac{g^{(j)}(a)}{j!}(x-a)^j\\
&= Q_n(x)
\end{aligned}$$

由於泰勒展式的前面 $n+1$ 項就是 $P_n(x)$, 由上面的證明可得

$$\begin{aligned}
& f(x) \sim a_0+a_1(x-a)+a_2(x-a)^2+\cdots+a_n(x-a)^n+\cdots\\
\Rightarrow\ & f'(x)=g(x) \sim a_1+2a_2(x-a)+\cdots+na_n(x-a)^{n-1}+\cdots
\end{aligned}$$

習題解答 4.2.5.

設

$$f(x) \sim a_0+a_1(x-a)+a_2(x-a)^2+a_3(x-a)^3\cdots+a_n(x-a)^n+\cdots$$

與

$$F(x) \sim b_0+b_1(x-a)+b_2(x-a)^2+b_3(x-a)^3\cdots+b_n(x-a)^n+\cdots$$

由上一習題知

$$(b_k(x-a)^k)'=k\cdot b_k(x-a)^{k-1}=a_{k-1}(x-a)^{k-1} \Rightarrow b_k=\frac{a_{k-1}}{k}, \quad k=1,2,\cdots$$

當 $k=0$ 時, $b_0=F(a)$, 所以證得

$$F(x) \sim F(a)+a_0(x-a)+\frac{a_1}{2}(x-a)^2+\frac{a_2}{3}(x-a)^3+\cdots+\frac{a_n}{n+1}(x-a)^{n+1}+\cdots$$

習題解答 4.2.10.

(2) $f''(x)\sim \frac{4}{5}+\frac{18}{10}x+\cdots+\frac{n^2(n-1)}{n^2+1}x^{n-2}+\cdots$

(5) $f(x^4)\sim 1+\frac{1}{2}x^4+\frac{2}{5}x^8+\cdots+\frac{n}{n^2+1}x^{4n}+\cdots$

(8) 由 (1) 知

$$f'(x)\sim \frac{1}{2}+\frac{4}{5}x+\cdots+\frac{n^2}{n^2+1}x^{n-1}+\cdots$$

所以

$$xf'(x)\sim \frac{1}{2}x+\frac{4}{5}x^2+\cdots+\frac{n^2}{n^2+1}x^n+\cdots$$

最後得

$$\int_0^x xf'(x)\,dx \sim \frac{1}{4}x^2+\frac{4}{15}x^3+\cdots+\frac{n^2}{(n+1)(n^2+1)}x^{n+1}+\cdots$$