3.4 積分的應用

習題解答 **3.4.6.**

$$\begin{aligned}\int_{r\cos\theta}^{y}\sqrt{r^2-x^2}\,dx &\overset{x=ru}{=} r^2\int_{\cos\theta}^{1}\sqrt{1-u^2}\,du\\ &= r^2\big(\frac{1}{2}(\sin^{-1}u+u\sqrt{1-u^2})\big)\big|_{\cos\theta}^{1}\\ &= \frac{r^2}{2}(\frac{\pi}{2}+0-(\frac{\pi}{2}-\theta)-\cos\theta\sin\theta)\\ &= \frac{r^2}{2}(\theta-\sin\theta\cos\theta)\end{aligned}$$

習題解答 **3.4.7.**

(3) $y=2\sin x\geq 2\sin x\cos x=\sin 2x$, 得

$$\begin{aligned}\int_0^{\pi}2\sin x-\sin 2x\;dx &= -2\cos x+\frac{\cos 2x}{2}\big|_0^{\pi}\\ &= 2+\frac{1}{2}-(-2)-\frac{1}{2}=4\end{aligned}$$

習題解答 **3.4.8.**

$$\begin{aligned}2\int_{-r}^{r}\sqrt{1+\big((\sqrt{r^2-x^2})'\big)^2}\,dx &= 2\int_{-r}^{r}\sqrt{1+\big(\frac{-x}{\sqrt{r^2-x^2}}\big)^2}\,dx\\ &= 2\int_{-r}^{r}\sqrt{1+\frac{x^2}{r^2-x^2}}\,dx\\ &= 2r\int_{-r}^{r}\frac{1}{\sqrt{r^2-x^2}}\,dx\\ &\overset{x=ru}{=} 2r\int_{-1}^{1}\frac{1}{\sqrt{1-u^2}}\cdot\frac{r}{r}\,du=2r\cdot\sin^{-1}u\big|_{-1}^{1}\\ &= 2r\cdot 2\frac{\pi}{2}=2\pi r\end{aligned}$$

習題解答 **3.4.9.**

(1) $y'=\frac{3}{2}x^{\frac{1}{2}}$, 則

$$\begin{aligned}\int_0^1\sqrt{1+(\frac{3}{2}x^{\frac{1}{2}})^2}\,dx &= \int_0^1\sqrt{1+\frac{9}{4}x}\,dx=\frac{2}{3}\cdot\frac{4}{9}(1+\frac{9}{4}x)^{\frac{3}{2}}\Big|_0^1\\ &= \frac{8}{27}((\frac{13}{4})^{\frac{3}{2}}-1)=\frac{13}{27}\sqrt{13}-\frac{8}{27}\end{aligned}$$

(4) $y'=\frac{e^x-e^{-x}}{2}$. 得

$$\begin{aligned}\int_{-1}^{1}\sqrt{1+(\frac{e^x-e^{-x}}{2})^2}\,dx &= \int_{-1}^{1}\sqrt{1+\frac{e^{2x}-2+e^{-2x}}{4}}\,dx\\ &= \int_{-1}^{1}\sqrt{(\frac{e^x+e^{-x}}{2})^2}\,dx=\int_{-1}^{1}\frac{e^x+e^{-x}}{2}\,dx\\ &= \frac{1}{2}(e^x-e^{-x})\big|_{-1}^{1}=\frac{1}{2}\cdot 2(e-e^{-1})=e-\frac{1}{e}\end{aligned}$$

習題解答 **3.4.14.**

(3) $\int_0^2\pi(x^2)^2\,dx=\pi\int_0^2 x^4\,dx=\pi\cdot\frac{x^5}{5}\big|_0^2=\frac{32}{5}\pi$

(6) 在此範圍中 $\sec x > \tan x$. 旋轉體體積為

$$\int_0^1 \pi(\sec^2 x - \tan^2 x)\ dx = \pi\int_0^1 1\ dx = \pi$$

習題解答 3.4.16.

將金字塔的底面放在水平面上, 設高為 t 時, 水平截面的面積 $A(t)$（其中 $0 \le t \le h$）. 利用圓盤法的想法, 金字塔體積等於

$$\lim_{N\to\infty}\sum_{i=1}^{N} A(t)\Delta t = \int_0^h A(t)\ dt$$

由金字塔的構造方式知道其水平截面為正方形, 設高為 t 時其邊長 $a(t)$, 可用比例計算得：

$$\frac{a}{h} = \frac{a(t)}{h-t} \quad\Rightarrow\quad a(t) = \frac{a}{h}(h-t)$$

因此在高為 t 時, 截面面積 $A(t) = a^2(t) = \frac{a^2}{h^2}(h-t)^2$. 因此金字塔體積等於

$$\int_0^h A(t)\ dt = \int_0^h \frac{a^2}{h^2}(h-t)^2\ dt = \frac{a^2}{h^2}\cdot\big(-\frac{(h-t^3)}{3}\big)\Big|_0^h = \frac{a^2}{h^2}\cdot\frac{h^3}{3} = \frac{1}{3}a^2h$$

習題解答 3.4.18.

(2) $2-x^2 = x^2 \Rightarrow x = \pm 1$. 用殼形法：

$$\begin{aligned}\int_0^1 2\pi x((2-x^2)-x^2)\ dx &= 2\pi\int_0^1 x(2-2x^2)\ dx = 4\pi\int_0^1 x - x^3\ dx \\ &= 4\pi\cdot\big(\frac{x^2}{2}-\frac{x^4}{4}\big)\Big|_0^1 = 4\pi(\frac{1}{2}-\frac{1}{4}) = \pi\end{aligned}$$

(4) 用殼形法：$3y^2-2 = y^2 \Rightarrow y = \pm 1$, 所以體積等於

$$\begin{aligned}\int_0^1 2\pi y(y^2-(3y^2-2))\ dy &= 2\pi\int_0^1 2y-2y^3))\ dy = 4\pi\cdot\big(\frac{y^2}{2}-\frac{y^4}{4}\big)\Big|_0^1 \\ &= 4\pi\cdot(\frac{1}{2}-\frac{1}{4}) = \pi\end{aligned}$$

習題解答 3.4.25.

由於各分量的重心定義和直線上的定義方式一樣, 因此滿足

$$M_1\cdot\bar{x}_1 + M_2\cdot\bar{x}_2 = M\cdot\bar{x}, \quad M_1\cdot\bar{y}_1 + M_2\cdot\bar{y}_2 = M\cdot\bar{y}$$

所以

$$\begin{aligned}& M_1\cdot(\bar{x}_1,\bar{y}_1) + M_2\cdot(\bar{x}_2,\bar{y}_2) \\ =\ & (M_1\cdot\bar{x}_1 + M_2\cdot\bar{x}_2,\ M_1\cdot\bar{y}_1 + M_2\cdot\bar{y}_2) \\ =\ & (M\cdot\bar{x},\ M\cdot\bar{y}) \\ =\ & M\cdot(\bar{x},\ \bar{y})\end{aligned}$$

習題解答 **3.4.26.**

檢視下列黎曼和

(1)

$$\begin{aligned}\sum_{i=1}^{N}\rho\cdot(f(\xi_i)-g(\xi_i))\frac{x_{i-1}+x_i}{2}\Delta x &= \sum_{i=1}^{N}\rho\cdot(f(\xi_i)-g(\xi_i))\frac{2x_{i-1}+\Delta x}{2}\Delta x\\ &\approx \sum_{i=1}^{N}\rho\cdot(f(\xi_i)-g(\xi_i))x_{i-1}\Delta x\end{aligned}$$

所以

$$\lim_{N\to\infty}\sum_{i=1}^{N}\rho\cdot(f(\xi_i)-g(\xi_i))\frac{x_{i-1}+x_i}{2}\Delta x=\rho\int_a^b x(f(x)-g(x))\,dx$$

(2)

$$\sum_{i=1}^{N}\rho\cdot(f(\xi_i)-g(\xi_i))\frac{f(\xi_i)+g(\xi_i)}{2}\Delta x=\sum_{i=1}^{N}\rho\cdot\frac{f^2(\xi_i)-g^2(\xi_i)}{2}\Delta x$$

所以

$$\lim_{N\to\infty}\sum_{i=1}^{N}\rho\cdot(f(\xi_i)-g(\xi_i))\frac{f(\xi_i)+g(\xi_i)}{2}\Delta x=\frac{\rho}{2}\int_a^b (f^2(x)-g^2(x))\,dx$$

(3) 另外

$$\lim_{N\to\infty}\sum_{i=1}^{N}\rho\cdot(f(\xi_i)-g(\xi_i))\Delta x=\rho\int_a^b (f(x)-g(x))\,dx$$

因此在消去分子和分母的 ρ 之後, 即證得欲證之重心公式.