3.3 基本積分技巧

習題解答 3.3.2.

$$\begin{aligned}\big((x^2-2x+2)e^x+C\big)' &= \big(x^2e^x-2xe^x+2e^x+C\big)' \\ &= 2xe^x+x^2e^x-(2e^x+2xe^x)+2e^x = x^2e^x\end{aligned}$$

習題解答 3.3.3.

(1) $\int x^3e^x\,dx$ 即 $E_3(x)$, 利用**例 3.3.2** 公式可得

$$\begin{aligned}E_3(x) &= x^3e^x-3E_2(x) \\ &= x^3e^x-3(x^2e^x-2(xe^x-e^x))+C \\ &= x^3e^x-3x^2e^x+6xe^x-6e^x+C\end{aligned}$$

習題解答 3.3.4.

(1)

$$\begin{aligned}&\left(\frac{1}{2}e^x(\sin x-\cos x)+C\right)' \\ =\ & (\frac{1}{2}e^x\sin x+\frac{1}{2}e^x\cos x)-(\frac{1}{2}e^x\cos x-\frac{1}{2}e^x\sin x) \\ =\ & e^x\sin x\end{aligned}$$

(2)

$$\begin{aligned}&\left(\frac{1}{2}e^x(\sin x+\cos x)+C\right)' \\ =\ & (\frac{1}{2}e^x\sin x+\frac{1}{2}e^x\cos x)+(\frac{1}{2}e^x\cos x-\frac{1}{2}e^x\sin x) \\ =\ & e^x\cos x\end{aligned}$$

$\boxed{\text{習題解答}}$ **3.3.5.**

(1)

$$\begin{array}{rcl}
\int e^{2x}\sin x\,dx &=& \int e^{2x}\,d(-\cos x) = -e^{2x}\cos x + 2\int e^{2x}\cos x\,dx \\
\| && \\
\int \sin x\,d(\frac{e^{2x}}{2}) &=& \frac{1}{2}e^{2x}\sin x - \frac{1}{2}\int e^{2x}\cos x\,dx \\
\Rightarrow\ \frac{5}{2}\int e^{2x}\cos x\,dx &=& \frac{1}{2}e^{2x}\sin x + e^{2x}\cos x + C \\
\Rightarrow\ \int e^{2x}\cos x\,dx &=& \big(\frac{1}{5}\sin x + \frac{2}{5}\cos x\big)e^{2x} + C
\end{array}$$

$$\begin{array}{rcl}
\int e^{2x}\cos x\,dx &=& \int e^{2x}\,d(\sin x) = e^{2x}\sin x - 2\int e^{2x}\sin x\,dx \\
\| && \\
\int \cos x\,d(\frac{e^{2x}}{2}) &=& \frac{1}{2}e^{2x}\cos x + \frac{1}{2}\int e^{2x}\sin x\,dx \\
\Rightarrow\ \frac{5}{2}\int e^{2x}\sin x\,dx &=& e^{2x}\sin x - \frac{1}{2}e^{2x}\cos x + C \\
\Rightarrow\ \int e^{2x}\sin x\,dx &=& \big(\frac{2}{5}\sin x - \frac{1}{5}\cos x\big)e^{2x} + C
\end{array}$$

(2)

$$\begin{array}{rcl}
\int e^{x}\sin 2x\,dx &=& \int e^{x}\,d(-\frac{\cos 2x}{2}) = -\frac{1}{2}e^{x}\cos 2x + \frac{1}{2}\int e^{x}\cos 2x\,dx \\
\| && \\
\int \sin 2x\,d(e^{x}) &=& e^{x}\sin 2x - 2\int e^{x}\cos 2x\,dx \\
\Rightarrow\ \frac{5}{2}\int e^{x}\cos 2x\,dx &=& e^{x}\sin 2x + \frac{1}{2}e^{x}\cos 2x + C \\
\Rightarrow\ \int e^{x}\cos 2x\,dx &=& \big(\frac{2}{5}\sin 2x + \frac{1}{5}\cos 2x\big)e^{x} + C
\end{array}$$

$$\begin{array}{rcl}
\int e^{x}\cos 2x\,dx &=& \int e^{2x}\,d(\frac{\sin 2x}{2}) = \frac{1}{2}e^{x}\sin 2x - \frac{1}{2}\int e^{x}\sin 2x\,dx \\
\| && \\
\int \cos 2x\,d(e^{x}) &=& e^{x}\cos 2x + 2\int e^{x}\sin 2x\,dx \\
\Rightarrow\ \frac{5}{2}\int e^{x}\sin 2x\,dx &=& \frac{1}{2}e^{2x}\sin x - e^{x}\cos 2x + C \\
\Rightarrow\ \int e^{x}\sin 2x\,dx &=& \big(\frac{1}{5}\sin 2x - \frac{2}{5}\cos 2x\big)e^{x} + C
\end{array}$$

$\boxed{\text{習題解答}}$ **3.3.6.**

(3)

$$\begin{array}{rcl}
\displaystyle\int x\ln x\,dx &=& \displaystyle\frac{1}{2}\int \ln x d(x^2) \\
&=& \displaystyle\frac{1}{2}(x^2\ln x - \int x^2\cdot\frac{1}{x}\,dx) = \frac{1}{2}x^2\ln x - \frac{1}{4}x^2 + C
\end{array}$$

(5)

$$\begin{array}{rcl}
\displaystyle\int x^2e^{-x}\,dx &=& \displaystyle -\int x^2d(e^{-x}) = -(x^2e^{-x} - \int e^{-x}2x\,dx) \\
&=& \displaystyle -(x^2e^{-x} + 2\int xd(e^{-x})) \\
&=& \displaystyle -[x^2e^{-x} + 2(xe^{-x} - \int e^{-x}\,dx)] \\
&=& -x^2e^{-x} - 2xe^{-x} - 2e^{-x} + C
\end{array}$$

習題解答 **3.3.9.**

(2)

$$\begin{aligned}\int \cos^{-1} x\, dx &\overset{x=\cos u}{=} -\int \cos^{-1}(\cos x)\cdot \sin u\, du \\ &= -\int u\sin u\, du = \int u d(\cos u) = u\cos u - \int \cos u\, du \\ &= u\cos u - \sin u + C = x\cos^{-1} x - \sqrt{1-x^2} + C\end{aligned}$$

習題解答 **3.3.10.**

(1) $\displaystyle\int x\sin(2x^2)\, dx \overset{u=2x^2}{=} \frac{1}{4}\int \sin u\, du = -\frac{1}{4}\cos u + C = -\frac{1}{4}\cos(2x^2) + C$

(5)

$$\begin{aligned}\int \frac{1}{x\sqrt{x^2-4}}\, dx &= \int \frac{1}{2x\sqrt{(\frac{x}{2})^2-1}}\, dx \overset{u=\frac{x}{2}}{=} \frac{1}{2}\int \frac{1}{u\sqrt{u^2-1}}\, du \\ &= \frac{1}{2}\sec^{-1} u + C = \frac{1}{2}\sec^{-1}(\frac{x}{2}) + C\end{aligned}$$

習題解答 **3.3.11.**

(2)

$$\begin{aligned}\int \sin(\ln x)\, dx &\overset{u=\ln x}{=} \int \sin u\cdot e^u\, du = \frac{1}{2}e^u(\sin u - \cos u) + C \\ &= \frac{x}{2}(\sin(\ln x) - \cos(\ln x)) + C\end{aligned}$$

(5) $\displaystyle\int x\sec^{-1}x\, dx = \int \sec^{-1}x\, d(\frac{x^2}{2}) = \frac{1}{2}x^2\sec^{-1}x - \frac{1}{2}\int x^2\cdot\frac{1}{|x|\sqrt{x^2-1}}\, dx$

當 $x>1$, 原式等於

$$\frac{1}{2}x^2\sec^{-1}x - \frac{1}{2}\int \frac{x}{\sqrt{x^2-1}}\, dx = \frac{x^2\sec^{-1}x - \sqrt{x^2-1}}{2} + C$$

當 $x<-1$, 原式等於

$$\frac{1}{2}x^2\sec^{-1}x + \frac{1}{2}\int \frac{x}{\sqrt{x^2-1}}\, dx = \frac{x^2\sec^{-1}x + \sqrt{x^2-1}}{2} + C$$

習題解答 **3.3.12.**

(3) $\displaystyle\int_0^1 x^3(1+x^4)^3\, dx \overset{u=1+x^4}{=} \frac{1}{4}\int_1^2 u^3\, du = \frac{1}{16}u^4\Big|_1^2 = \frac{15}{16}$

$\boxed{\text{習題解答}}$ **3.3.16.**

(3) 設

$$\frac{1}{x^2(x^2-1)} = \frac{1}{x^2(x-1)(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x-1)} + \frac{D}{(x+1)}$$

通分得

$$1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1)$$

代值得

$$\begin{aligned} x=0 &\quad\Rightarrow\quad B=-1 \\ x=1 &\quad\Rightarrow\quad 1=2C \Rightarrow C=\frac{1}{2} \\ x=-1 &\quad\Rightarrow\quad 1=-2D \Rightarrow D=-\frac{1}{2} \\ A+C+D=0 &\quad\Rightarrow\quad A=0 \end{aligned}$$

故原式

$$\begin{aligned} \int \frac{1}{x^2(x^2-1)}\,dx &= \int \frac{-1}{x^2} + \frac{1}{2(x-1)} - \frac{1}{2(x+1)}\,dx \\ &= \frac{1}{x} + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C \end{aligned}$$

(6)

$$\begin{aligned} \int \frac{x^2}{(1+x^2)^2}\,dx &= \int \frac{1+x^2-1}{(1+x^2)^2}\,dx = \int \frac{1}{1+x^2}\,dx - \int \frac{1}{(1+x^2)^2}\,dx \\ &= \tan^{-1}x - \left(\frac{1}{2}\tan^{-1}x + \frac{1}{2}\frac{x}{1+x^2}\right) + C \\ &= \frac{1}{2}\tan^{-1}x - \frac{1}{2}\frac{x}{1+x^2} + C \end{aligned}$$

也可用下面的作法：

$$\begin{aligned} \int \frac{x^2}{(1+x^2)^2}\,dx &\overset{x=\tan\theta}{=} \int \frac{\tan^2\theta}{\sec^4\theta}\sec^2\theta\,d\theta = \int \sin^2\theta\,d\theta \\ &= \int \frac{1-\cos 2\theta}{2}\,d\theta = \frac{\theta}{2} - \frac{1}{4}\sin 2\theta + C \\ &= \frac{\theta}{2} - \frac{1}{2}\sin\theta\cos\theta + C \\ &\overset{x=\tan\theta}{=} \frac{1}{2}\tan^{-1}x - \frac{1}{2}\frac{x}{1+x^2} + C \end{aligned}$$

(9)

$$\int \frac{e^x}{e^{2x}+3e^x+2}\,dx \overset{u=e^x}{=} \int \frac{1}{u^2+3u+2}\,du$$

設

$$\frac{1}{u^2+3u+2} = \frac{1}{(u+1)(u+2)} = \frac{A}{(u+1)} + \frac{B}{(u+2)}$$

通分得

$$1 = A(u+2) + B(u+1)$$

代值得

$$\begin{aligned} u=-2 &\quad\Rightarrow\quad B=-1 \\ u=1 &\quad\Rightarrow\quad A=1 \end{aligned}$$

原式

$$\begin{aligned} \int \frac{1}{u^2+3u+2}\,du &= \int \frac{1}{(u+1)} - \frac{1}{(u+2)}\,du \\ &= \ln\left|\frac{u+1}{u+2}\right| + C \overset{u=e^x}{=} \ln\left|\frac{e^x+1}{e^x+2}\right| + C \end{aligned}$$

習題解答 3.3.17.

$$\begin{aligned} \int \sin^m x\cos^{2k+1} x\,dx &= \int \sin^m x(\cos^2 x)^k \cdot \cos x\,dx \\ &= \int \sin^m x(1-\sin^2 x)^k\,d(\sin x) \\ &\overset{u=\sin x}{=} \int u^m(1-u^2)^k\,du \end{aligned}$$

習題解答 3.3.19.

(1)

$$\begin{aligned} \int \sqrt{1-x^2}\,dx &\overset{x=\sin\theta}{=} \int \cos^2\theta\,d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta \\ &= \frac{\theta}{2} + \frac{1}{4}\sin 2\theta + C \\ &\overset{x=\sin\theta}{=} \frac{1}{2}\sin^{-1}x + \frac{1}{2}x\sqrt{1-x^2} + C \end{aligned}$$

(4)

$$\begin{aligned} \int \frac{1}{x\sqrt{1-x^2}}\,dx &\overset{x=\sin\theta}{=} \int \frac{1}{\sin\theta\cos\theta} \cdot \cos\theta\,d\theta \\ &= -\ln|\csc\theta + \cot\theta| + C = -\ln\left|\frac{1}{x} + \sqrt{\frac{1}{x^2}-1}\right| + C \end{aligned}$$

(7)

$$\begin{aligned} \int \frac{\sqrt{x^2+1}}{x}\,dx &\overset{x=\tan\theta}{=} \int \frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int \frac{\sec^2\theta}{\tan^2\theta} \cdot \tan\theta\sec\theta\,d\theta \\ &= \int \left(1+\frac{1}{\tan^2\theta}\right) \cdot \tan\theta\sec\theta\,d\theta \\ &= \int \tan\theta\sec\theta\,d\theta + \int \csc\theta\,d\theta = \sec\theta - \ln|csc\theta + \cot\theta| + C \\ &\overset{x=\tan\theta}{=} \sqrt{1+x^2} - \ln\left|\frac{1+\sqrt{1+x^2}}{x}\right| + C \\ \Big(&= \sqrt{1+x^2} - \ln\left|1+\sqrt{1+x^2}\right| + \ln|x| + C \quad\Big) \end{aligned}$$

習題解答 3.3.20.

(1)

$$
\begin{aligned}
& \int \sqrt{1+x+x^2}\, dx \\
= \quad & \int \sqrt{(x+\frac{1}{2})^2+\frac{3}{4}}\, dx \\
= \quad & \int \sqrt{\frac{3}{4}\cdot\Big(\frac{4}{3}(x+\frac{1}{2})^2+1\Big)}\, dx \\
= \quad & \frac{\sqrt{3}}{2}\int \sqrt{\big(\frac{2}{\sqrt{3}}(x+\frac{1}{2})\big)^2+1}\, dx \\
\overset{u=\frac{2}{\sqrt{3}}(x+\frac{1}{2})}{=} \quad & \frac{\sqrt{3}}{2}\int \sqrt{u^2+1}\cdot\frac{\sqrt{3}}{2}\, du \\
= \quad & \frac{3}{4}\Big(\frac{1}{2}(u\sqrt{u^2+1}+\ln(u+\sqrt{u^2+1}))\Big)+C \\
\overset{u=\frac{2}{\sqrt{3}}(x+\frac{1}{2})}{=} \quad & \frac{3}{8}\Big(\frac{2}{\sqrt{3}}(x+\frac{1}{2})\cdot\frac{2}{\sqrt{3}}\sqrt{x^2+x+1} \\
& +\ln\big(\frac{2}{\sqrt{3}}(x+\frac{1}{2})+\frac{2}{\sqrt{3}}\sqrt{x^2+x+1}\big)\Big)+C \\
= \quad & \frac{1}{2}(x+\frac{1}{2})\sqrt{x^2+x+1}+\frac{3}{8}\ln(x+\frac{1}{2}+\sqrt{x^2+x+1})+C
\end{aligned}
$$

(5)

$$
\begin{aligned}
\int \frac{1}{x^2\sqrt{x^2+1}}\, dx & \overset{x=\tan\theta}{=} \int \frac{1}{\tan^2\theta\sec\theta}\sec^2\theta\, d\theta = \int \frac{1}{\sin^2\theta}\cos\theta\, d\theta \\
& \overset{u=\sin\theta}{=} \int \frac{1}{u^2}\, du = -\frac{1}{u}+C \overset{u=\frac{x}{\sqrt{1+x^2}}}{=} -\frac{\sqrt{1+x^2}}{x}+C
\end{aligned}
$$

習題解答 3.3.21.

(1)

$$
\begin{aligned}
S_n(x) &= \int \sin^n x\, dx = \int \sin^{n-1} x\, d(-\cos x) \\
&= -\cos x\sin^{n-1} x+(n-1)\int (1-\sin^2 x)\sin^{n-2} x\, dx \\
&= -\cos x\sin^{n-1} x+(n-1)\int \sin^{n-2} x-\sin^n x\, dx \\
&= -\cos x\sin^{n-1} x+(n-1)S_{n-2}(x)-(n-1)S_n(x) \\
\Rightarrow \quad S_n(x) &= -\frac{1}{n}\cos x\sin^{n-1} x+\frac{n-1}{n}S_{n-2}(x)
\end{aligned}
$$

(2) 已知 $S_0(x)=\int 1\, dx=x+C$, $S_1(x)=\int \sin x\, dx=-\cos x+C$, 因此

$$
\begin{aligned}
\int \sin^2 x\, dx = S_2(x) &= -\frac{1}{2}\cos x\sin x+\frac{1}{2}S_0(x) = -\frac{1}{2}\cos x\sin x+\frac{1}{2}x+C \\
\int \sin^3 x\, dx = S_3(x) &= -\frac{1}{3}\cos x\sin^2 x+\frac{2}{3}S_1(x) \\
&= -\frac{1}{3}\cos x\sin^2 x-\frac{2}{3}\cos x+C \\
\int \sin^4 x\, dx = S_4(x) &= -\frac{1}{4}\cos x\sin^3 x+\frac{3}{4}S_2(x) \\
&= -\frac{1}{4}\cos x\sin^3 x+\frac{3}{4}(-\frac{1}{2}\cos x\sin x+\frac{1}{2}x)+C \\
&= -\frac{1}{4}\cos x\sin^3 x-\frac{3}{8}\cos x\sin x+\frac{3}{8}x+C
\end{aligned}
$$

$\boxed{\text{習題解答}}$ **3.3.22.**

(1) 直接計算如下

$$\begin{aligned}
\int_0^{2\pi} \cos^k x\, dx &= \int_0^{2\pi} \cos^{k-1} x\, d(\sin x) \\
&= \sin x\ \cos^{k-1} x\Big|_0^{2\pi} - \int_0^{2\pi} \sin x \cdot (\cos^{k-1} x)'\, dx \\
&= 0 + (k-1)\cdot \int_0^{2\pi} \sin^2 x \cos^{k-2} x\, dx \\
&= (k-1)\cdot \int_0^{2\pi} (1-\cos^2 x)\cos^{k-2} x\, dx \\
&= (k-1)\int_0^{2\pi} \cos^{k-2} x\, dx - (k-1)\int_0^{2\pi} \cos^k x\, dx
\end{aligned}$$

移項合併整理得 $\displaystyle\int_0^{2\pi} \cos^k x\, dx = \frac{k-1}{k}\int_0^{2\pi} \cos^{k-2} x\, dx,\ k \geq 2.$

(2) 由 (1) 可得

$$\begin{aligned}
\int_0^{2\pi} \cos^{2k} x\, dx &= \frac{2k-1}{2k}\cdot \int_0^{2\pi} \cos^{2k-2} x\, dx \\
&= \frac{(2k-1)\cdot(2k-3)}{2k\cdot(2k-2)}\cdot \int_0^{2\pi} \cos^{2k-4} x\, dx = \cdots \\
&= \frac{(2k-1)\cdot(2k-3)\cdots 3\cdot 1}{2k\cdot(2k-2)\cdots 4\cdot 2}\cdot \int_0^{2\pi} 1\, dx \\
&= \frac{(2k)!}{(2k\cdot(2k-2)\cdots 4\cdot 2)^2}\cdot 2\pi = \frac{(2k)!\,\pi}{2^{2k-1}(k!)^2} \\
\int_0^{2\pi} \cos^{2k+1} x\, dx &= \frac{2k}{2k+1}\cdot \int_0^{2\pi} \cos^{2k-1} x\, dx = \cdots \\
&= \frac{2k\cdot(2k-2)\cdots 4\cdot 2}{(2k+1)\cdot(2k-1)\cdots 5\cdot 3}\cdot \int_0^{2\pi} \cos x\, dx = 0
\end{aligned}$$

$\boxed{\text{習題解答}}$ **3.3.23.**

$$\begin{aligned}
\int \frac{1}{(1+x^2)^3}\, dx &\overset{x=\tan\theta}{=} \int \frac{1}{\sec^6\theta}\sec^2\theta\, d\theta = \int \cos^4\theta\, d\theta \\
&= \int (\cos\theta)^2\, d\theta \ = \int \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta \\
&= \int \left(\frac{1}{4}+\frac{\cos 2\theta}{2}\right) d\theta + \frac{1}{4}\int \cos^2 2\theta\, d\theta \\
&= \frac{\theta}{4} + \frac{\sin 2\theta}{4} + \frac{1}{4}\int \left(\frac{1+\cos 4\theta}{2}\right) d\theta \\
&= \frac{3}{8}\theta + \frac{\sin 2\theta}{4} + \frac{1}{8}\cdot\frac{\sin 4\theta}{4} + C \\
&\overset{x=\tan\theta}{=} \frac{3}{8}\tan^{-1} x + \frac{x}{2(x^2+1)} + \frac{1}{8}\frac{x(1-x^2)}{(1+x^2)^2} + C
\end{aligned}$$

以上用到

$$\sin 4\theta = 2\sin 2\theta\cos 2\theta = 4\cos^4\theta\cdot\tan\theta(1-\tan^2\theta) = 4\frac{\tan\theta(1-\tan^2\theta)}{\sec^4\theta}$$