3.2 微積分基本定理

習題解答 **3.2.1.**

$\int_0^\pi \sin x\ dx = \lim_{N\to\infty} \sum_{i=1}^{N} \sin \xi_i \Delta x$, 取 $\xi_i = \frac{x_{i-1}+x_i}{2}$（中點），
則 $\xi_i - \frac{\Delta x}{2} = x_{i-1}$, $\xi_i + \frac{\Delta x}{2} = x_i$.

$$\begin{aligned}\lim_{N\to\infty} \sum_{i=1}^{N} \sin \xi_i \Delta x &= \lim_{N\to\infty} \sum_{i=1}^{N} \frac{\cos(\xi_i - \frac{\Delta x}{2}) - \cos(\xi_i + \frac{\Delta x}{2})}{2\sin\frac{\Delta x}{2}} \Delta x \\ &= \lim_{N\to\infty} \sum_{i=1}^{N} \frac{\cos x_{i-1} - \cos x_i}{2\sin\frac{\Delta x}{2}} \Delta x \\ &= \lim_{N\to\infty} (\cos 0 - \cos\pi) \cdot \lim_{\Delta x\to 0} \frac{\frac{\Delta x}{2}}{\sin\frac{\Delta x}{2}} \\ &= 2\cdot 1 = 2\end{aligned}$$

習題解答 **3.2.2.**

(1) $\int_a^b 2\ dx = 2x\big|_a^b = 2(b-a)$

(2) $\int_a^b x\ dx = \frac{x^2}{2}\big|_a^b = \frac{1}{2}(b^2-a^2)$

(3) $\int_a^b x^3\ dx = \frac{x^4}{4}\big|_a^b = \frac{1}{4}(b^4-a^4)$

(4) $\int_0^1 \sqrt{x}\ dx = \int_0^1 x^{\frac{1}{2}}\ dx = \frac{2}{3}x^{\frac{3}{2}}\big|_0^1 = \frac{2}{3}$

(5) $\int_1^2 \frac{1}{x}\ dx = \ln|x|\big|_1^2 = \ln 2 - 0 = \ln 2$

(6) $\int_1^2 \frac{1}{x^2}\ dx = \int_1^2 x^{-2}\ dx = -x^{-1}\big|_1^2 = -\frac{1}{2}+1 = \frac{1}{2}$

(7) $\int_0^\pi \cos x\ dx = \sin x\big|_0^\pi = 0-0 = 0$

(8) $\int_{-1}^1 \frac{1}{1+x^2}\ dx = \tan^{-1} x\big|_{-1}^1 = \frac{\pi}{4} - (-\frac{\pi}{4}) = \frac{\pi}{2}$

(9) $\int_0^1 \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x\big|_0^1 = \frac{\pi}{2} - 0 = \frac{\pi}{2}$

習題解答 **3.2.3.**

令 $F(x) = \int_a^x f(t)\ dt$, 則 $F'(x) = f(x)$. 又

$$\int_{h(x)}^{g(x)} f(t)\ dt = \int_a^{g(x)} f(t)\ dt - \int_a^{h(x)} f(t)\ dt = F(g(x)) - F(h(x))$$

所以

$$\begin{aligned}\frac{d}{dx}(\int_{h(x)}^{g(x)} f(t)\ dt) &= (F(g(x)) - F(h(x)))' \\ &= (F(g(x)))' - (F(h(x)))' \\ &= F'(g(x))\cdot g'(x) - F'(h(x))\cdot h'(x) \\ &= f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x)\end{aligned}$$

習題解答 **3.2.4.**

(1)

$$\begin{aligned}\Big(\int_{\frac{x^2}{2}}^{x^2} \ln\sqrt{t}\ dt\Big)' &= \ln\sqrt{x^2}\cdot 2x - \ln\sqrt{\frac{x^2}{2}}\cdot x \\ &= 2x\ln x - (\ln x - \frac{1}{2}\ln 2)\cdot x = x\ln x + \frac{1}{2}\ln 2\end{aligned}$$

(2) $\Big(\int_1^{\sin x} 3t^2\ dt\Big)' = 3\sin^2 x\cdot\cos x.$

(3) $\Big(\int_0^{\tan x} \frac{1}{1+t^2}\ dt\Big)' = \frac{1}{1+\tan^2 x}\cdot\sec^2 x\frac{1}{\sec^2 x}\cdot\sec^2 x = 1.$

(4) $\Big(\int_0^{\tan^{-1}x} \sec^2 t\ dt\Big)' = (\sec\tan^{-1}x)^2\cdot\frac{1}{1+x^2} = (\sqrt{1+x^2})^2\cdot\frac{1}{1+x^2} = 1.$

(5)

$$\begin{aligned}\Big(\int_{-\sqrt{x}}^{\sqrt{x}} \sin(t^2)\ dt\Big)' &= \sin(\sqrt{x}^2)\cdot\frac{1}{2\sqrt{x}} - \sin(-\sqrt{x})^2\cdot(-\frac{1}{2\sqrt{x}}) \\ &= \sin x\cdot\frac{1}{2\sqrt{x}} - \sin x\cdot(-\frac{1}{2\sqrt{x}}) = \frac{\sin x}{\sqrt{x}}\end{aligned}$$

(6)

$$\begin{aligned}\Big(\int_{-\sqrt{x}}^{\sqrt{x}} \sin(t^3)\ dt\Big)' &= \sin(\sqrt{x})^3\cdot\frac{1}{2\sqrt{x}} - \sin(-\sqrt{x})^3\cdot\frac{-1}{2\sqrt{x}} \\ &= \sin(\sqrt{x})^3\cdot\frac{1}{2\sqrt{x}} - \sin(\sqrt{x})^3\cdot\frac{1}{2\sqrt{x}} = 0\end{aligned}$$

註 $\sin t^3$ 是奇函數, 積分範圍對稱於原點, 所以原定積分已等於 0.