2.1 導函數

習題解答 2.1.3.

相當於說明邊際成本不等於邊際收益, 則淨利不可能最大: 假設如果邊際成本大於邊際收益, 則稍微降低成本, 總淨利會越多；如果邊際成本少於邊際收益, 則稍微增加成本, 總淨利還會更多；所以這都表示此時的總淨利不是最大.

習題解答 2.1.4.

因為斜率是無窮大, 故可定義為 0. 這表示需求量固定, 和價格無關. 譬如某人特別喜歡某種茶飲料, 不管價格多少都會喝. 因此價格的變化對他的需求意願毫無影響, 所以此飲料對這個人的需求彈性是 0.

習題解答 2.1.7.

$$
\begin{aligned}
(\frac{f(x)}{g(x)})' &= (f(x)\cdot\frac{1}{g(x)})' = f'(x)\cdot\frac{1}{g(x)} + f(x)\cdot\frac{-g'(x)}{(g(x))^2} \\
&= \frac{f'(x)g(x)-f(x)g'(x)}{(g(x))^2}
\end{aligned}
$$

習題解答 2.1.9.

(1) $y' = (x^2)' = 2x$, 在 $x=1$ 的切線方程式:

$$y = y'(1)(x-1)+y(1) = 2(x-1)+1 = 2x-1$$

(2) $y' = (x^3)' = 3x^2$, 在 $x=1$ 的切線方程式:

$$y = y'(1)(x-1)+y(1) = 3(x-1)+1 = 3x-2$$

(3) $y' = (x+\frac{1}{x})' = 1+(\frac{1}{x})' = 1-\frac{1}{x^2}$, 在 $x=1$ 的切線方程式:

$$y = y'(1)(x-1)+y(1) = 0+2 = 2$$

習題解答 2.1.10.

$$
\begin{aligned}
v(t) &= s'(t) = 0 + (0 + v_0) - \frac{1}{2}g(t^2)' = v_0 - gt \\
a(t) &= v'(t) = 0 - (0 + g) = -g
\end{aligned}
$$

習題解答 2.1.11.

(1) 當 $n = 1$ 時, $((f(x)))' = 1 \cdot (f(x))^0 \cdot f'(x) = f'(x)$, 原式成立.

(2) 設當 $n = k$ 時, 原式成立, 即 $((f(x))^k)' = k \cdot (f(x))^{k-1} \cdot f'(x)$.

(3) 當 $n = k + 1$ 時,

$$
\begin{aligned}
\text{左式} &= ((f(x))^{k+1})' \\
&= ((f(x))^k \cdot f(x))' \\
&= ((f(x))^k)' \cdot f(x) + (f(x))^k \cdot f'(x) \\
&= k \cdot (f(x))^{k-1} \cdot f'(x) \cdot f(x) + (f(x))^k \cdot f'(x) \\
&= k \cdot (f(x))^k \cdot f'(x) + (f(x))^k \cdot f'(x) \\
&= (k+1) \cdot f(x))^k \cdot f'(x) \\
&= \text{右式}
\end{aligned}
$$

故得證.

習題解答 2.1.14.

因為 $(t)^2 = (t^2)$, 所以 (t^2, t) 是 $y^2 = x$ 的參數式. 計算得 $(t^2, t)' = (2t, 1)$.

(1) 此曲線在 $t = -2$ 的切線方向向量為 $(-4, 1)$, 切線方程式為

$$
\begin{aligned}
y - y(-2) = \frac{1}{-4}(x - x(-2)) \quad &\Rightarrow \quad y + 2 = -\frac{1}{4}(x - 4) \\
&\Rightarrow \quad y = -\frac{1}{4}x - 1
\end{aligned}
$$

(2) 此曲線在 $t = 0$ 的切線方向向量為 $(0, 1)$, 切線方向垂直於 x-軸, 故方程式為

$$
x = x(0) \quad \Rightarrow \quad x = 0
$$

習題解答 2.1.15.

$$
\begin{aligned}
& (a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0)' \\
= \; & (a_n x^n)' + \cdots + (a_2 x^2)' + (a_1 x)' + (a_0)' \\
= \; & a_n (x^n)' + \cdots + a_2 (x^2)' + a_1 (x)' + 0 \\
= \; & n \cdot a_n x^{n-1} + \cdots + 2a_2 x + a_1
\end{aligned}
$$

習題解答 **2.1.18.**

(1) 利用 $(x^{\frac{1}{3}})^3 = x$, 則

$$((x^{\frac{1}{3}})^3)' = x' \;\Rightarrow\; 3\cdot(x^{\frac{1}{3}})^2\cdot(x^{\frac{1}{3}})' = 1 \;\Rightarrow\; (x^{\frac{1}{3}})' = \frac{1}{3\cdot(x^{\frac{1}{3}})^2} = \frac{1}{3}x^{-\frac{2}{3}}$$

(2) 利用 $(x^{\frac{2}{3}})^3 = x^2$, 則

$$((x^{\frac{2}{3}})^3)' = (x^2)' \;\Rightarrow\; 3\cdot(x^{\frac{2}{3}})^2\cdot(x^{\frac{2}{3}})' = 2x \;\Rightarrow\; (x^{\frac{2}{3}})' = \frac{2x}{3\cdot(x^{\frac{2}{3}})^2} = \frac{2}{3}x^{-\frac{1}{3}}$$

習題解答 **2.1.19.**

(1) $(\tan x)' = (\dfrac{\sin x}{\cos x})' = \dfrac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} = \dfrac{\sin^2 x + \cos^2 x}{\cos^2 x} = \sec^2 x$

(2) $(\cot x)' = (\dfrac{\cos x}{\sin x})' = \dfrac{(\cos x)'\sin x - \cos x(\sin x)'}{\sin^2 x} = \dfrac{-(\sin^2 x + \cos^2 x)}{\sin^2 x} = -\csc^2 x$

(3) $(\sec x)' = (\dfrac{1}{\cos x})' = \dfrac{1'\cdot\cos x - 1\cdot(\cos x)'}{\cos^2 x} = \dfrac{-(-\sin x)}{\cos^2 x} = \sec x\tan x$

(4) $(\csc x)' = (\dfrac{1}{\sin x})' = \dfrac{1'\cdot\sin x - 1\cdot(\sin x)'}{\sin^2 x} = \dfrac{-\cos x}{\sin^2 x} = -\csc x\cot x$

習題解答 **2.1.21.**

(2) $(\dfrac{x^2-x+1}{x^2+x+1})' = \dfrac{(2x-1)(x^2+x+1)-(2x+1)(x^2-x+1)}{(x^2+x+1)^2} = \dfrac{2(x^2-1)}{(x^2+x+1)^2}$

(4) $(\sin x\cdot\ln x)' = (\sin x)'\cdot\ln x + \sin x\cdot(\ln x)' = \cos x\cdot\ln x + \dfrac{\sin x}{x}$

(6)

$$\begin{aligned}(\frac{\log_2 x}{x^2})' &= (\frac{\frac{\ln x}{\ln 2}}{x^2})' = \frac{1}{\ln 2}\cdot\frac{(\ln x)'x^2 - \ln x(x^2)'}{(x^2)^2}\\ &= \frac{1}{\ln 2}\cdot\frac{\frac{x^2}{x} - 2x\ln x}{x^4}\\ &= \frac{1}{\ln 2}\cdot\frac{1-2\ln x}{x^3} \qquad (\text{或 } \frac{\frac{1}{\ln 2} - 2\log_2 x}{x^3})\end{aligned}$$

習題解答 **2.1.23.**

利用 $(x^{\frac{q}{p}})^p = x^q$, 則

$$((x^{\frac{q}{p}})^p)' = (x^q)' \Rightarrow p\cdot(x^{\frac{q}{p}})^{p-1}\cdot(x^{\frac{q}{p}})' = qx^{q-1} \Rightarrow (x^{\frac{q}{p}})' = \frac{qx^{q-1}}{p\cdot(x^{\frac{q}{p}})^{p-1}} = \frac{q}{p}x^{\frac{q}{p}-1}$$

習題解答 **2.1.27.**

$$\begin{aligned}(\ln ax)' &= (\ln a + \ln x)' = \frac{1}{x}\\ (\ln x^n)' &= (n\ln x)' = \frac{n}{x}\end{aligned}$$

習題解答 **2.1.30.**

設 $g(x) = x^n$, 則 $g'(x) = nx^{n-1}$

$$(f(x)^n)' = (g(f(x)))' = g'(f(x))\cdot f'(x) = n(f(x))^{n-1}\cdot f'(x)$$

得證.

習題解答 **2.1.32.**

$$(\tan^{-1}x)' = \frac{1}{\tan'(\tan^{-1}x)} = \frac{1}{(\sec(\tan^{-1}x))^2} = \frac{1}{(\sqrt{1+x^2})^2} = \frac{1}{1+x^2}$$

因為 $\cot x^{-1} = \frac{\pi}{2} - \tan^{-1}x$, $(\cot x^{-1})' = -(\tan^{-1}x)' = -\frac{1}{1+x^2}$.

$$(\sec^{-1}x)' = \frac{1}{\sec'(\sec^{-1}x)} = \frac{1}{\sec(\sec^{-1}x)\cdot\tan(\sec^{-1}x)}$$

由第一章習題知

$$\frac{1}{\sec(\sec^{-1}x)\cdot\tan(\sec^{-1}x)} = \begin{cases} \frac{1}{x\sqrt{x^2-1}}, & x \geq 1 \\ \frac{1}{x(-\sqrt{x^2-1})}, & x \leq -1 \end{cases}$$

因此

$$\frac{1}{\sec(\sec^{-1}x)\cdot\tan(\sec^{-1}x)} = \frac{1}{|x|\sqrt{x^2-1}}$$

習題解答 **2.1.34.**

(2) $(\sqrt{(x^2+x+1)})' = \frac{1}{2}\cdot(x^2+x+1)^{\frac{-1}{2}}\cdot(2x+1)$

(4) $(\sin(2^x))' = \cos(2^x)\cdot(2^x)' = \cos(2^x)\cdot 2^x\cdot\ln 2$

(6) $(e^{\sin^{-1}x})' = e^{\sin^{-1}x}\cdot(\sin^{-1}x)' = e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}}$

(8)

$$\begin{aligned}\left(\sin(\frac{x}{\sqrt{x+1}})\right)' &= \cos\frac{x}{\sqrt{x+1}}\cdot(\frac{x}{\sqrt{x+1}})' \\ &= \cos(\frac{x}{\sqrt{x+1}})\cdot\frac{(x+1)^{\frac{1}{2}} - \frac{1}{2}x(x+1)^{\frac{-1}{2}}}{x+1} \\ &= \cos(\frac{x}{\sqrt{x+1}})\cdot(x+1)^{\frac{-3}{2}}\cdot(1+\frac{x}{2})\end{aligned}$$

習題解答 **2.1.35.**

$$\begin{aligned}(f(x)^{g(x)})' &= (e^{g(x)\ln f(x)})' \\ &= (g(x)\ln f(x))' \cdot (e^{g(x)\ln f(x)}) \\ &= \left(g'(x)\ln f(x) + g(x)\cdot \frac{f'(x)}{f(x)}\right)\cdot f(x)^{g(x)} \\ &= \ln f(x)\cdot f(x)^{g(x)}\cdot g'(x) + g(x)\cdot f(x)^{g(x)-1}\cdot f'(x)\end{aligned}$$

習題解答 **2.1.36.**

(1) $(x^x)' = x\cdot x^{x-1}\cdot (x)' + \ln x\cdot x^x\cdot (x)' = x^x + \ln x\cdot x^x = (1+\ln x)x^x$

(2)

$$\begin{aligned}((\sin x)^{\cos x})' &= \cos x\cdot(\sin x)^{\cos x-1}(\sin x)' + \ln\sin x\cdot(\sin x)^{\cos x}\cdot(\cos x)' \\ &= \cos^2 x\cdot(\sin x)^{\cos x-1} - \ln\sin x\cdot(\sin x)^{\cos x+1}\end{aligned}$$

(3)

$$\begin{aligned}(x^{\ln x})' &= \ln x\cdot x^{\ln x-1}\cdot(x)' + \ln x\cdot x^{\ln x}\cdot(\ln x)' \\ &= \ln x\cdot x^{\ln x-1} + \ln x\cdot x^{\ln x}\cdot \tfrac{1}{x} \\ &= 2\ln x\cdot x^{\ln x-1}\end{aligned}$$

習題解答 **2.1.38.**

$$(\ln f(x))' = \frac{f'(x)}{f(x)} \quad\Rightarrow\quad f'(x) = f(x)\cdot(\ln f(x))'$$

但

$$\ln f(x) = m_1\ln f_1(x) + m_2\ln f_2(x) + \cdots + m_k\ln f_k(x) = \sum_{i=1}^{k}(m_i\cdot\ln f_i(x))$$

所以

$$f'(x) = f(x)\cdot(\sum_{i=1}^{k}(m_i\cdot\ln f_i(x)))' = f(x)\cdot(\sum_{i=1}^{k}(m_i\cdot\frac{f_i'(x)}{f_i(x)}))$$

習題解答 **2.1.40.**

(1) 因為 $f(x)$ 在 $x = x(t_0) = a$ 的切線斜率為 $f'(a)$, 因此切線之方向向量可取為 $(1, f'(a))$, 由前面習題知：

$$(x'(t_0), y'(t_0)) = \lambda\cdot(1, f'(a))$$

(2) 由上習題知有某 λ, 使得 $(x'(t_0), y'(t_0)) = (\lambda, \lambda f'(a))$, 於是

$$f'(a) = \frac{\lambda f'(a)}{\lambda} = \frac{y'(t_0)}{x'(t_0)}$$

.

(3) 函數曲線 $y = f(x)$ 有參數式 $(x(t), y(t))$, 這表示 $y(t) = f(x(t))$, 由連鎖法則得

$$y'(t) = (f(x(t)))' = f'(x(t)) \cdot x'(t)$$

代 $t = t_0$ 與 $x(t_0) = a$, 得

$$y'(t_0) = f'(x(t_0)) \cdot x'(t_0) \quad \Rightarrow \quad f'(a) = f'(x(t_0)) = \frac{y'(t_0)}{x'(t_0)}$$

習題解答 2.1.41.

(3) $x(t) = \dfrac{e^t + e^{-t}}{2}$, $x'(t) = \dfrac{e^t - e^{-t}}{2}$; $y(t) = \dfrac{e^t - e^{-t}}{2}$, $y'(t) = \dfrac{e^t + e^{-t}}{2}$, 代 $t = 1$ 得

$$y - \frac{e - \frac{1}{e}}{2} = \frac{\frac{e+\frac{1}{e}}{2}}{\frac{e-\frac{1}{e}}{2}}(x - \frac{e + \frac{1}{e}}{2}) \quad \Rightarrow \quad y - \frac{e - \frac{1}{e}}{2} = \frac{e^2+1}{e^2-1}(x - \frac{e + \frac{1}{e}}{2})$$

(5) $x(t) = \tan t$, $x'(t) = \sec^2 t$; $y(t) = \sec t$, $y'(t) = \sec t \tan t$, 代 $t = \frac{\pi}{4}$ 得

$$y - \sqrt{2} = \frac{1}{\sqrt{2}}(x - 1)$$

(6) $x(t) = \sin^{-1} t$, $y(t) = \cos^{-1} t$, 此為直線 $x + y = \frac{\pi}{2}$ 的參數式, 因此對任何 t 值, 其切線都是 $x + y = \frac{\pi}{2}$.

習題解答 2.1.42.

取單位圓之參數式為 $(\cos t, \sin t)$, 即 $x(t) = \cos t$, $x'(t) = -\sin t$; $y(t) = \sin t$, $y'(t) = \cos t$, 令 $x(t_0) = x_0$, $y(t_0) = y_0$, 若 $y_0 \neq 0$, 則切線為

$$\begin{aligned} y - y_0 = \frac{x_0}{-y_0}(x - x_0) \quad &\Rightarrow \quad -y_0 y + (y_0)^2 = x_0 x - (x_0)^2 \\ &\Rightarrow \quad x_0 x + y_0 y = (x_0)^2 + (y_0)^2 = 1 \end{aligned}$$

當 $y_0 = 0$ 時, $x_0 = \pm 1$, 切線垂直於 x-軸, 即 $x = \pm 1$, 也滿足 $1x + 0y = 1$ 或 $(-1)x + 0y = 1$ 的形式.

習題解答 2.1.43.

(5) $(\ln(2^x x^2))'' = (\ln 2 + \dfrac{2}{x})' = -\dfrac{2}{x^2}$

(6)

$$\begin{aligned} \left(\frac{\log_2 x}{x^2}\right)'' &= \left(\frac{1}{\ln 2} \cdot \frac{1 - 2\ln x}{x^3}\right)' = \frac{1}{\ln 2} \cdot \frac{\frac{-2}{x} \cdot x^3 - (1 - 2\ln x) \cdot 3x^2}{x^6} \\ &= \frac{1}{\ln 2} \cdot \frac{-5 + 6\ln x}{x^4} \qquad \left(\text{或 } \frac{\frac{-5}{\ln 2} + 6\log_2 x}{x^4}\right) \end{aligned}$$

習題解答 2.1.44.

(2)

$$\begin{aligned}(\sqrt{(x^2+x+1)})'' &= \left(\frac{1}{2}\cdot\frac{2x+1}{(x^2+x+1)^{\frac{1}{2}}}\right)' \\ &= \frac{1}{2}\cdot\frac{2(x^2+x+1)^{\frac{1}{2}}-(2x+1)\cdot\frac{1}{2}(x^2+x+1)^{\frac{-1}{2}}\cdot(2x+1)}{x^2+x+1} \\ &= \frac{3}{4}\cdot\frac{1}{(x^2+x+1)^{\frac{3}{2}}}\end{aligned}$$

(4)

$$\begin{aligned}(\sin(2^x))'' &= (\cos(2^x)\cdot 2^x\cdot\ln 2)' \\ &= \ln 2(-\sin(2^x)\cdot\ln 2\cdot(2^x)^2+\cos(2^x)\cdot\ln 2\cdot 2^x) \\ &= (\ln 2)^2\cdot 2^x\cdot(\cos(2^x)-\sin(2^x)\cdot 2^x)\end{aligned}$$

(6)

$$\begin{aligned}(e^{\sin^{-1}x})'' &= (e^{\sin^{-1}x}\cdot\frac{1}{\sqrt{1-x^2}})' \\ &= e^{\sin^{-1}x}\cdot(\frac{1}{\sqrt{1-x^2}})^2+e^{\sin^{-1}x}\cdot\frac{-1}{2}\cdot\frac{1}{(1-x^2)^{\frac{3}{2}}}\cdot(-2x) \\ &= e^{\sin^{-1}x}\left(\frac{1}{1-x^2}+\frac{x}{(1-x^2)^{\frac{3}{2}}}\right)\end{aligned}$$

(8)

$$\begin{aligned}\left(\sin(\frac{x}{\sqrt{x+1}})\right)'' &= (\cos(\frac{x}{\sqrt{x+1}})\cdot(x+1)^{\frac{-3}{2}}\cdot(1+\frac{x}{2}))' \\ &= -\sin(\frac{x}{\sqrt{x+1}})\cdot((x+1)^{\frac{-3}{2}}\cdot(1+\frac{x}{2}))^2 \\ &\quad +\cos(\frac{x}{\sqrt{x+1}})(\frac{-3}{2}(x+1)^{\frac{-5}{2}}\cdot(1+\frac{x}{2})+(x+1)^{\frac{-3}{2}}\cdot\frac{1}{2}) \\ &= -\sin(\frac{x}{\sqrt{x+1}})(\frac{1+x+\frac{x^2}{4}}{(x+1)^3}) \\ &\quad -\cos(\frac{x}{\sqrt{x+1}})\cdot(1+\frac{x}{4})\cdot(x+1)^{\frac{-5}{2}}\end{aligned}$$

習題解答 2.1.48.

函數曲線 $y=f(x)$ 有參數式 $(x(t),y(t))$, 這表示 $y(t)=f(x(t))$, 由連鎖法則得

$$y'(t)=(f(x(t)))'=f'(x(t))\cdot x'(t)$$

繼續在等式兩邊微分

$$y''(t)=(f'(x(t))\cdot x'(t))'=f''(x(t))\cdot(x'(t))^2+f'(x(t))\cdot x''(t)$$

代 $t=t_0$ 與 $x(t_0)=a$, 得

$$y''(t_0)=f''((a))\cdot(x'(t_0))^2+f'(a)\cdot x''(t_0)\quad\Rightarrow\quad f''(a)=\frac{y''(t_0)-f'(a)\cdot x''(t_0)}{(x'(t_0))^2}$$

但已知

$$f'(a)=\frac{y'(t_0)}{x'(t_0)}$$

代入得

$$f''(a)=\frac{y''(t_0)-\frac{y'(t_0)}{x'(t_0)}\cdot x''(t_0)}{(x'(t_0))^2}=\frac{x'(t_0)y''(t_0)-x''(t_0)y'(t_0)}{(x'(t_0))^3}$$

習題解答 2.1.50.

(2)

$$\begin{aligned} & (y-\ln y)' = (-x+\ln x+1)' \\ \Rightarrow\quad & y' - \tfrac{y'}{y} = -1 + \tfrac{1}{x} \\ \Rightarrow\quad & y' = \frac{-1+\frac{1}{x}}{1-\frac{1}{y}} = -\frac{y(x-1)}{x(y-1)} \end{aligned}$$

(4)

$$\begin{aligned} & (\sin(xy))' = (x+y)' \\ \Rightarrow\quad & \cos(xy)\cdot(y+xy') = 1+y' \\ \Rightarrow\quad & y' = -\frac{1-y\cos xy}{1-x\cos xy} \end{aligned}$$

習題解答 2.1.51.

(1) 對等號兩邊微分

$$(x^2+xy-y^2)' = 1' \Rightarrow 2x+y+xy'-2yy' = 0$$

將 $x=2$, $y=3$ 代入得

$$4+3+2y'(2)-6y'(2)=0 \quad\Rightarrow\quad y'(2)=\frac{7}{4}$$

將第一式再微分一次得

$$(2x+y+xy'-2yy')' = 0 \Rightarrow 2+y'+y'+xy''-2(y')^2-2yy''=0$$

將 $x=2$, $y=3$, $y'(2)=\frac{7}{4}$ 代入得

$$2+2\cdot\frac{7}{4}+2y''(2)-2\cdot(\frac{7}{4})^2-6y''(2)=0 \quad\Rightarrow\quad y''(2)=-\frac{5}{32}$$

(2) 對等號兩邊微分

$$(2xy+\pi\sin y) = (2\pi)' \Rightarrow 2y+2xy'+\pi\cos y\cdot y' = 0$$

將 $x=1$, $y=\frac{\pi}{2}$ 代入得

$$\pi+2y'(1)+0=0 \quad\Rightarrow\quad y'(1)=-\frac{\pi}{2}$$

將第一式再微分一次得

$$\begin{aligned} & (2y+2xy'+\pi\cos y\cdot y')' = 0 \\ \Rightarrow\quad & 2y'+2y'+2xy''-\pi\sin y\cdot(y')^2+\pi\cos y\cdot y'' = 0 \end{aligned}$$

將 $x=1$, $y=\frac{\pi}{2}$, $y'(1)=-\frac{\pi}{2}$ 代入得

$$4\cdot(-\frac{\pi}{2})+2y''(1)-\pi\cdot(\frac{\pi}{2})^2+0=0 \quad\Rightarrow\quad y''(1)=\frac{2\pi+\frac{\pi^3}{4}}{2}=\frac{8\pi+\pi^3}{8}$$

(3) 對等號兩邊微分

$$(\ln(x-y))' = (\frac{x}{y}-1)' \Rightarrow \frac{1-y'}{x-y} = \frac{1}{y} - \frac{xy'}{y^2}$$

將 $x=2e,\ y=e$ 代入得

$$\frac{1-y'(2e)}{e} = \frac{1}{e} - \frac{2e\,y'(2e)}{e^2} \quad \Rightarrow \quad y'(2e)=0$$

將第一式再微分一次得

$$\begin{aligned} &(\frac{1-y'}{x-y})' = (\frac{1}{y} - \frac{xy'}{y^2})' \\ \Rightarrow \quad & \frac{-y''(x-y)-(1-y')^2}{(x-y)^2} = -\frac{y'}{y^2} - \frac{(y'+xy'')y^2 - xy'\cdot 2yy'}{y^4} \end{aligned}$$

將 $x=2e,\ y=e,\ y'(2e)=0$ 代入得

$$\frac{-ey''(2e)-1}{e^2} = 0 - \frac{2e^3y''(2e)-0}{e^4} \quad \Rightarrow \quad y''(2e) = \frac{1}{e}$$

(4) 對等號兩邊微分

$$(\tan^{-1}\frac{y}{x})' = (2xy+y^2)' \Rightarrow \frac{1}{1+\frac{y^2}{x^2}}\cdot(\frac{y'}{x}-\frac{y}{x^2}) = 2y+2xy'+2yy'$$

將 $x=1,\ y=0$ 代入得

$$1\cdot(y'(1)-0) = 0+2y'(1)+0 \quad \Rightarrow \quad y'(1)=0$$

將第一式再微分一次得

$$\begin{aligned} &(\frac{1}{1+\frac{y^2}{x^2}}\cdot(\frac{y'}{x}-\frac{y}{x^2}))' = (2y+2xy'+2yy')' \\ \Rightarrow \quad & (\frac{xy'-y}{x^2+y^2})' = (2y+2xy'+2yy')' \\ \Rightarrow \quad & \frac{(y'+xy''-y')(x^2+y^2)-(xy'-y)(2x+2yy')}{(x^2+y^2)^2} \\ & = 2(y'+y'+xy''+(y')^2+yy'') \end{aligned}$$

將 $x=1,\ y=0,\ y'(1)=0$ 代入得

$$\frac{y''(1)\cdot 1-0}{1^2} = 0+2y''(1)+0 \quad \Rightarrow \quad y''(1)=0$$