1.4 連續函數與極限

習題解答 1.4.1.

如果此數列有兩個相異的極限 L_1 和 L_2, 令 $\epsilon = \frac{|L_1-L_2|}{4} \neq 0$. 依照課本極限的定義, 這數列終究會落到 $(L_1-\epsilon, L_1+\epsilon)$ 內, 同時也終究會落到 $(L_2-\epsilon, L_2+\epsilon)$ 內. 但是這兩個範圍根本不相交, 這違反了數列的「終究」定義, 因此數列不可能有兩個以上的極限.

習題解答 1.4.2.

(1) 由題意知：a_n 終究會等於 $L+$(很小的數)；b_n 終究會等於 $M+$(很小的數). 所以

$$a_n \cdot b_n = (L + (\text{很小的數})) \cdot (M + (\text{很小的數})) = L \cdot M + (\text{很小的數})$$

因此 $\lim\limits_{n\to\infty}(a_n \cdot b_n) = L \cdot M$.

(2) 由題意知：b_n 終究會等於 $M+$(很小的數). 所以

$$\begin{aligned}\frac{1}{b_n} &= \frac{1}{M + (\text{很小的數})} = \frac{1}{M(1 + (\text{很小的數}))} \\ &= \frac{1}{M} \cdot (1 + (\text{很小的數})) = \frac{1}{M} + (\text{很小的數})\end{aligned}$$

因此 $\lim\limits_{n\to\infty} \frac{1}{b_n} = \frac{1}{M}$.

習題解答 1.4.5.

(1) $l > k$:

$$\begin{aligned}&\lim_{n\to\infty} \frac{a_k n^k + a_{k-1} n^{k-1} + \cdots + a_0}{b_l n^l + b_{l-1} n^{l-1} + \cdots + b_0} \\ =& \frac{\lim\limits_{n\to\infty}(a_k \frac{1}{n^{l-k}} + a_{k-1} \frac{1}{n^{l-k+1}} + \cdots + a_0 \frac{1}{n^l})}{\lim\limits_{n\to\infty}(b_l + b_{l-1}\frac{1}{n} + \cdots + b_0 \frac{1}{n^l})} \\ =& \frac{0 + 0 + \cdots + 0}{b_l + 0 + \cdots + 0} = 0\end{aligned}$$

(2) $l = k$:

$$\begin{aligned}
&\lim_{n\to\infty} \frac{a_k n^k + a_{k-1}n^{k-1} + \cdots + a_0}{b_k n^k + b_{k-1}n^{k-1} + \cdots + b_0} \\
= &\frac{\lim\limits_{n\to\infty}(a_k + a_{k-1}\frac{1}{n} + \cdots + a_0\frac{1}{n^k})}{\lim\limits_{n\to\infty}(b_k + b_{k-1}\frac{1}{n} + \cdots + b_0\frac{1}{n^k})} \\
= &\frac{a_k + 0 + \cdots + 0}{b_k + 0 + \cdots + 0} = \frac{a_k}{b_k}
\end{aligned}$$

(3) $k > l$

$$\begin{aligned}
&\lim_{n\to\infty} \frac{a_k n^k + a_{k-1}n^{k-1} + \cdots + a_0}{b_l n^l + b_{l-1}n^{l-1} + \cdots + b_0} \\
= &\frac{\lim\limits_{n\to\infty}(a_k n^{k-l} + a_{k-1}n^{k-l-1} + \cdots + a_0\frac{1}{n^l})}{\lim\limits_{n\to\infty}(b_l + b_{l-1}\frac{1}{n} + \cdots + b_0\frac{1}{n^l})} \\
= &\frac{(\lim\limits_{n\to\infty} a_k n^{k-l}) + \cdots + a_l + 0 + \cdots + 0}{b_l + 0 + \cdots + 0} = \pm\infty
\end{aligned}$$

習題解答 1.4.6.

(1) 因為 $S_n = \sum_{k=1}^{n} n = \frac{n^2+n}{2}$, 所以 $\sum_{k=1}^{\infty} n = \lim\limits_{n\to\infty} \frac{n^2+n}{2} = \infty$.

(2) 因為 $S_n = \sum_{k=1}^{n} n^2 = \frac{2n^3+3n^2+n}{6}$, 所以 $\sum_{k=1}^{\infty} n^2 = \lim\limits_{n\to\infty} \frac{2n^3+3n^2+n}{6} = \infty$.

(3) $S_n = \sum_{k=1}^{n}(-1)^n$, 計算可知

$$S_n = \begin{cases} -1, & n \text{ 是奇數} \\ 0, & n \text{ 是偶數} \end{cases}$$

所以 $\sum_{k=1}^{\infty}(-1)^n = \lim\limits_{n\to\infty} S_n$ 無極限.

(4)

$$S_n = \sum_{k=1}^{n} \frac{1}{2^k} = \frac{1}{2} \cdot \frac{1 - (\frac{1}{2})^n}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n}$$

所以 $\sum_{k=1}^{\infty} \frac{1}{2^k} = \lim\limits_{n\to\infty} (1 - \frac{1}{2^n}) = 1$.

習題解答 1.4.7.

假設 $f(x)$ 和 $g(x)$ 是連續函數.

$$\begin{cases} \lim\limits_{x\to a}(f(x)\pm g(x)) = \lim\limits_{x\to a} f(x) \pm \lim\limits_{x\to a} g(x) = f(a)\pm g(a) \\ \lim\limits_{x\to a}(f(x)\cdot g(x)) = \lim\limits_{x\to a} f(x)\cdot \lim\limits_{x\to a} g(x) = f(a)\cdot g(a) \\ \lim\limits_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)} = \dfrac{f(a)}{g(a)} \\ \lim\limits_{x\to a} f(g(x)) = f(\lim\limits_{x\to a} g(x)) = f(g(a)) \end{cases}$$

所以 $f(x)\pm g(x)$, $f(x)\cdot g(x)$, $\dfrac{f(x)}{g(x)}$, $f(g(x))$ 是連續函數.

習題解答 1.4.8.

(1)
$$\lim_{x\to a}\frac{x^3-a^3}{x-a} = \lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)}{x-a} = \lim_{x\to a}(x^2+ax+a^2) = 3a^2$$

(2)
$$\begin{aligned}\lim_{x\to a}\frac{x^{-2}-a^{-2}}{x-a} &= \lim_{x\to a}\frac{\frac{a^2-x^2}{a^2\cdot x^2}}{x-a} = \lim_{x\to a}\left(-\frac{1}{a^2\cdot x^2}\frac{(x-a)(x+a)}{x-a}\right) \\ &= \lim_{x\to a} -\frac{x+a}{a^2\cdot x^2} = -\frac{2}{a^3}\end{aligned}$$

(3) $n>0$.

$$\begin{aligned}\lim_{x\to a}\frac{x^n-a^n}{x-a} &= \lim_{x\to a}\frac{(x-a)(x^{n-1}+ax^{n-2}+\cdots+a^{n-2}x+a^{n-1})}{x-a} \\ &= \lim_{x\to a}(x^{n-1}+ax^{n-2}+\cdots+a^{n-2}x+a^{n-1}) = na^{n-1}\end{aligned}$$

$n=-m<0$, $a\neq 0$.

$$\begin{aligned}&\lim_{x\to a}\frac{x^{-m}-a^{-m}}{x-a} = \lim_{x\to a}\frac{\frac{a^m-x^m}{a^m\cdot x^m}}{x-a} \\ =& \lim_{x\to a}\left(-\frac{1}{a^m\cdot x^m}\cdot\frac{(x-a)(x^{m-1}+ax^{m-2}+\cdots+a^{m-2}x+a^{m-1})}{x-a}\right) \\ =& \lim_{x\to a} -\frac{x^{m-1}+ax^{m-2}+\cdots+a^{m-2}x+a^{m-1}}{a^m\cdot x^m} = -\frac{m}{a^{m+1}} = na^{n-1}\end{aligned}$$

都是 na^{n-1}

習題解答 1.4.9.

(2) $\lim\limits_{x\to 1}\dfrac{x^2-3x+1}{2x-1} = \dfrac{\lim\limits_{x\to 1} x^2-3x+1}{\lim\limits_{x\to 1} 2x-1} = \dfrac{-1}{1} = -1$

(5) $\lim\limits_{x\to \frac{4}{\pi}}\log(\tan\dfrac{1}{x}) = \log(\lim\limits_{x\to \frac{4}{\pi}}\tan\dfrac{1}{x}) = \log(\tan(\lim\limits_{x\to \frac{4}{\pi}}\dfrac{1}{x})) = \log(\tan\dfrac{\pi}{4}) = \log 1 = 0$

(6) 先計算

$$\lim_{x\to 1}\sin(2\tan^{-1}x) = \sin(\lim_{x\to 1} 2\tan^{-1}x) = \sin(2\tan^{-1}1) = \sin(2\cdot\frac{\pi}{4}) = 1$$

所以

$$\lim_{x\to 1} 2^{\sin(2\tan^{-1}x)} = 2^{\lim\limits_{x\to 1}\sin(2\tan^{-1}x)} = 2^1 = 2$$

習題解答 1.4.10.

(1) $\lim\limits_{x\to 1}\dfrac{x^3-1}{x^4-1}=\lim\limits_{x\to 1}\dfrac{(x-1)(x^2+x+1)}{(x-1)(x^3+x^2+x+1)}=\lim\limits_{x\to 1}\dfrac{x^2+x+1}{x^3+x^2+x+1}=\dfrac{3}{4}$

(4)

$$\begin{aligned}\lim_{x\to -2}\frac{\sqrt{x^3+x^2+8}+x}{x+2} &= \lim_{x\to -2}\frac{(\sqrt{x^3+x^2+8}+x)(\sqrt{x^3+x^2+8}-x)}{(x+2)(\sqrt{x^3+x^2+8}-x)}\\ &= \lim_{x\to -2}\frac{x^3+8}{(x+2)(\sqrt{x^3+x^2+8}-x)}\\ &= \lim_{x\to -2}\frac{(x+2)(x^2-2x+4)}{(x+2)(\sqrt{x^3+x^2+8}-x)}\\ &= \lim_{x\to -2}\frac{x^2-2x+4}{\sqrt{x^3+x^2+8}-x}=\frac{12}{4}=3\end{aligned}$$

(5) $\lim\limits_{x\to 0}\dfrac{\tan x}{\sin 2x}=\lim\limits_{x\to 0}\dfrac{\frac{\sin x}{\cos x}}{2\sin x\cos x}=\lim\limits_{x\to 0}\dfrac{1}{2\cos^2 x}=\dfrac{1}{2}$

習題解答 1.4.12.

(3) $\lim\limits_{x\to\infty}\dfrac{x^3-1}{x^4+1}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{x}-\frac{1}{x^4}}{1+\frac{1}{x^4}}=0$

(4) 因為當 x 很大時必是正數, 因此

$$\frac{-1}{x}\le\frac{\sin x}{x}\le\frac{1}{x}$$

故

$$\lim_{x\to\infty}\frac{-1}{x}\le\lim_{x\to\infty}\frac{\sin x}{x}\le\lim_{x\to\infty}\frac{1}{x}$$

由夾擊法得 $\lim\limits_{x\to\infty}\dfrac{\sin x}{x}=0.$

(5) $\lim\limits_{x\to\infty}\tan^{-1}x=\dfrac{\pi}{2}$

(6) $\lim\limits_{x\to\infty}\sin(\tan^{-1}x)=\sin(\lim\limits_{x\to\infty}\tan^{-1}x)=\sin\dfrac{\pi}{2}=1$

習題解答 1.4.13.

同第四節下述習題結果：

$$\lim_{n\to\infty}\frac{a_kn^k+a_{k-1}n^{k-1}+\cdots+a_0}{b_ln^l+b_{l-1}n^{l-1}+\cdots+b_0}$$

習題解答 1.4.14.

因為

$$\lim_{x\to\pm\infty}\frac{P(x)}{Q(x)}=\lim_{x\to\pm\infty}(\text{多項式}+\frac{R(x)}{Q(x)})$$

但因為 $R(x)$ 的次數小於 $Q(x)$ 的次數, $\lim\limits_{x\to\pm\infty}\frac{R(x)}{Q(x)}=0$. 所以當 $x\to\pm\infty$ 時, 函數圖形漸近於多項式的部分.

習題解答 1.4.15.

如果當 $f(x)$ 隨著 x 趨近 ∞, 終究會落在 L 的附近, 其中整數點 $x=n$ 所對應的 $f(n)=a_n$ 當然終究會落在 L 的附近.