微乙小考五 (2017/6/8)

1. (8分) 定義隨機變數

$$X = \begin{cases} 2 \\ 1 \end{cases} ; \quad \begin{cases} P(X=2) = \frac{1}{2} \\ P(X=1) = \frac{1}{2} \end{cases}$$

$$Y = \begin{cases} 1 \\ -1 \end{cases} ; \quad \begin{cases} P(Y=1) = \frac{2}{3} \\ P(Y=-1) = \frac{1}{3} \end{cases}$$

求

(a) $E(3X+2Y)$

(b) 假設 X, Y 獨立，求 $Var(3X+2Y)$

sol: (a)

$$E(3X+2Y) = 3E(X) + 2E(Y) = 3\left[2 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}\right] + 2\left[1 \cdot \frac{2}{3} + (-1) \cdot \frac{1}{3}\right]$$
$$= \frac{9}{2} + \frac{2}{3} = \frac{31}{6} \qquad \square$$

(b)

$$Var(3X+2Y) = Var(3X) + Var(2Y) = 9Var(X) + 4Var(Y)$$
$$= 9\left[\left(4 \cdot \frac{1}{2} + 1 \cdot \frac{1}{2}\right) - \left(\frac{3}{2}\right)^2\right] + 4\left[\left(1 \cdot \frac{2}{3} + 1 \cdot \frac{1}{3}\right) - \left(\frac{1}{3}\right)^2\right] = \frac{209}{36} \qquad \square$$

2. (5分) 求 $\int_0^\infty \frac{e^{-2x}}{\sqrt{x}} dx$.

sol: Let $u = \sqrt{2x}$, then $u^2 = 2x \implies 2u\mathrm{d}u = 2\mathrm{d}x$

Then we have $\int_0^\infty \frac{e^{-u^2}}{u/\sqrt{2}} \cdot u\mathrm{d}u = \sqrt{2}\int_0^\infty e^{-u^2}\mathrm{d}u = \sqrt{2} \cdot \frac{\sqrt{\pi}}{2} = \sqrt{\frac{\pi}{2}}$ $\square$

3. (7分) 給定機率密度函數 $f_X(t) = \begin{cases} e^{-t} & t \geq 0 \\ 0 & t < 0 \end{cases}$。令 $Y = 3X - 1$，求

(a) $F_Y(t)$

(b) $f_Y(t)$

sol: $F_Y(t) = P(Y \leq t) = P(3X - 1 \leq t) = P(X \leq \frac{t+1}{3})$

$$= \begin{cases} \int_0^{\frac{t+1}{3}} e^{-s}\mathrm{d}s & \text{if } \frac{t+1}{3} \geq 0 \\ 0 & \text{if } \frac{t+1}{3} < 0 \end{cases} = \begin{cases} -e^{-s}\Big|_0^{\frac{t+1}{3}} & \text{if } t \geq -1 \\ 0 & \text{if } t < -1 \end{cases} = \begin{cases} 1 - e^{-\frac{t+1}{3}} & \text{if } t \geq -1 \\ 0 & \text{if } t < -1 \end{cases}$$ Thus,

$$f_Y(t) = \frac{\mathrm{d}}{\mathrm{d}t}F_Y(t) = \begin{cases} \frac{1}{3}e^{-\frac{t+1}{3}} & \text{if } t \geq -1 \\ 0 & \text{if } t < -1 \end{cases} \qquad \square$$