微乙小考四 (2017/5/11)

1. (10分) 解微分方程 $\begin{cases} y' = -\dfrac{3}{t}y + \dfrac{e^t}{t^2} \\ y(1) = 2 \end{cases}$.

sol:

$$y' + \frac{3}{t}y = \frac{e^t}{t^2}$$

Integrating factor: $e^{\int \frac{3}{t}dt} = e^{3\ln t} = t^3, t > 0$

Therefore,

$$t^3y' + 3t^2y = te^t$$
$$\left(t^3y\right)' = te^t$$
$$\int \left(t^3y\right)' dt = \int te^t dt$$
$$t^3y = (t-1)e^t + c$$
$$\because y(1) = 2 \Rightarrow c = 2$$
$$\therefore y(t) = \frac{(t-1)e^t + 2}{t^3}$$

2. (10分) 設某一地區人口數之函數為 $P(t)$，且人口數之變化滿足 $\dfrac{dP(t)}{dt} = \lambda P(t) - b$，其中 $\begin{cases} \lambda = 1\% \\ b = 300000 \end{cases}$。若 2015 年時，人口數有 $P(0) = 23000000$，問於 2035 年時，人口數 $P(20)$ 為何?

sol:

$$\frac{dP}{dt} = \lambda P - b$$
$$\frac{dP}{dt} - \lambda P = -b$$

Integrating factor: $e^{\int -\lambda dt} = e^{-\lambda t}$

Therefore,

$$e^{-\lambda t}\frac{dP}{dt} - \lambda e^{-\lambda t}P = -be^{-\lambda t}$$
$$\left(e^{-\lambda t}P\right)' = -be^{-\lambda t}$$
$$e^{-\lambda t}P = \frac{b}{\lambda}e^{-\lambda t} + c$$
$$P(t) = \frac{b}{\lambda} + ce^{\lambda t}$$
$$\because P(0) = 2.3 \times 10^7 \Rightarrow c = -0.7 \times 10^7$$
$$\therefore P(20) = (3 - 0.7e^{\frac{1}{5}}) \times 10^7$$