1. (11%) Let $\mathbf{F} = z\cos(xz)\mathbf{i} + ze^{yz}\mathbf{j} + (x\cos(xz) + ye^{yz})\mathbf{k}$.
   (a) (8%) Find a scalar function $\varphi(x,y,z)$ such that $\nabla\varphi = \mathbf{F}$.
   (b) (3%) Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $C$ is the curve $\mathbf{r}(t) = (\cos(\pi t^2), \ln(t+1), \tan^{-1}(t)), 0 \le t \le 1$.

**Solution:**

(a) Because

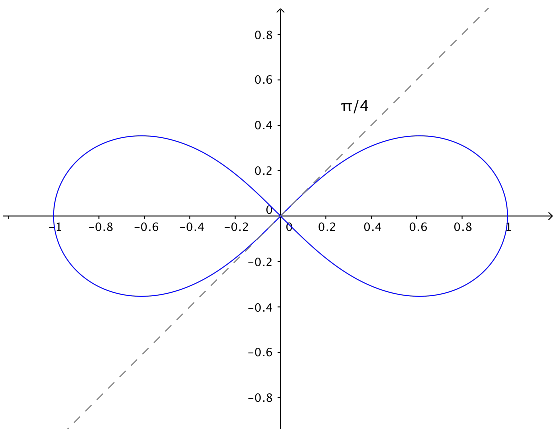
$$\begin{aligned} &\varphi_x(x,y,z) = z\cos(xz) && \Rightarrow \varphi(x,y,z) = \sin(xz) + g_1(y,z) \\ &\varphi_y(x,y,z) = ze^{yz} && \Rightarrow \varphi(x,y,z) = e^{yz} + g_2(x,z) \\ &\varphi_z(x,y,z) = x\cos(xz) + ye^{yz} && \Rightarrow \varphi(x,y,z) = \sin(xz) + e^{yz} + g_3(x,y), \end{aligned}$$

we can conclude that $\varphi(x,y,z) = \sin(xz) + e^{yz} + \text{const}$.
• If $\varphi$ is written as a vector but the above three calculations are right, you lose 3pts.

(b) Because $\mathbf{F}$ is conservative,

$$\begin{aligned} \int_C \mathbf{F}\cdot\, d\mathbf{r} &= \varphi(\mathbf{r}(1)) - \varphi(\mathbf{r}(0)) \\ &= \varphi(-1, \ln 2, \pi/4) - \varphi(1,0,0) \\ &= -\frac{1}{\sqrt{2}} + 2^{\pi/4} - 1. \end{aligned}$$

2. (12%) Let $C$ be the polar curve defined by $r^2 = \cos 2\theta$ in the first quadrant. Evaluate $\int_C y\, ds$.

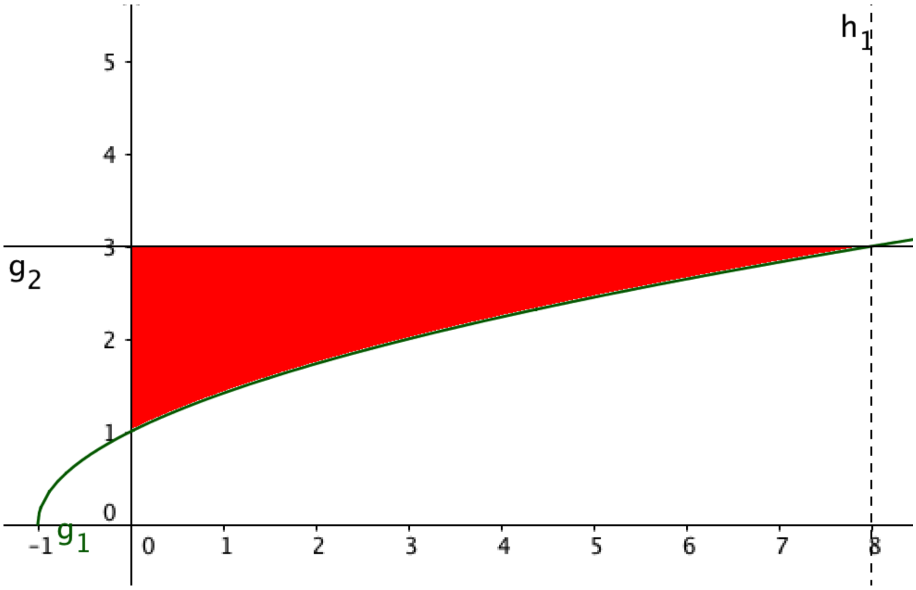



**Solution:**
Let $x = r\cos\theta$, $y = r\sin\theta$ (2pt)

$$\begin{aligned} \int_C y ds &= \int_0^{\frac{\pi}{4}} r\sin\theta\sqrt{r^2 + \left(\tfrac{dr}{d\theta}\right)^2} d\theta \quad \text{(4pt)} \\ &= \int_0^{\frac{\pi}{4}} r\sin\theta\sqrt{\cos 2\theta + \left(\tfrac{-\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2} d\theta \\ &= \int_0^{\frac{\pi}{4}} r\sin\theta\tfrac{1}{\sqrt{\cos 2\theta}} d\theta \quad \text{(4pt)} \\ &= \int_0^{\frac{\pi}{4}} \sin\theta d\theta = -\cos\theta\Big|_0^{\frac{\pi}{4}} = 1 - \tfrac{\sqrt{2}}{2} \quad \text{(2pt)} \end{aligned}$$

3. (11%) Sketch the region of integration and evaluate the integral $\displaystyle\int_0^8 \int_{\sqrt{1+x}}^3 \cos\left(\frac{x}{y+1}\right) dydx$.

**Solution:**




(4 points)

$$\int_0^8 \int_{\sqrt{1+x}}^3 cos(\frac{x}{y+1})\, dy\, dx$$
$$= \int_1^3 \int_0^{y^2-1} cos(\frac{x}{y+1})\, dx\, dy$$ (4 points)
$$= \int_1^3 \left[(y+1)\, sin(\frac{x}{y+1})\right]_0^{y^2-1} dy$$
$$= \int_1^3 (y+1)\, [sin(y-1)]\, dy$$

Let $u = y+1$, $dv = sin(y-1)dy$, and by applying Integration by parts,

$$\int_1^3 (y+1)\, [sin(y-1)]\, dy$$
$$= \left[-(y+1)cos(y-1)\right]_1^3 + \left[sin(y-1)\right]_1^3$$ (2 points)
$$= -4cos(2) + 2 + sin(2) - 0$$ (1 points)
$$= 2 + sin(2) - 4cos(2)$$

Note:

1. If you sketch the wrong part of the region, you will gain (2 points) for sketching part.
2. If you write $\int_1^3 \int_{y^2-1}^8 \cdots$ instead of the correct answer, you will gain (2 points).
3. If you write $\int_0^3 \int_{y^2-1}^8 \cdots$ instead of the correct answer, you will gain (0 point).
4. If you miss the minus in Integration by parts, you will gain (1 point).

4. (12%) Evaluate $\iint_R e^{\frac{x+2y}{2x-y}}\, dA$, where $R$ is the region in the $xy$-plane bounded by the four lines $2x-y=1$, $2x-y=2$,

$x - 3y = 0$ and $3x - 4y = 0$.

**Solution:**

Method 1:

Let $u = x + 2y$ ; $v = 2x - y$, then we can get $|J| = |\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|| = \dfrac{1}{5}$

Therefore, $\iint_R e^{\frac{x+2y}{2x-y}} dA = \int_1^2 \int_v^{2v} e^{\frac{u}{v}} |J| du dv = \int_1^2 \int_v^{2v} e^{\frac{u}{v}} \frac{1}{5} du dv = \frac{1}{5} \int_1^2 v e^{\frac{u}{v}} |_{u=v}^{u=2v} = \frac{1}{5} \int_1^2 v(e^2 - e) = \frac{3}{10}(e^2 - e)$.

Method 2:

Let $u = 2x - y$ ; $v = x - 3y$, then we can get $|J| = |\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|| = \dfrac{1}{5}$

Therefore, $\iint_R e^{\frac{x+2y}{2x-y}} dA = \int_1^2 \int_{-u}^0 e^{\frac{u-v}{u}} |J| dv du = \int_1^2 \int_{-u}^0 e^{\frac{-v}{u}} e \frac{1}{5} dv du = \frac{3}{10}(e^2 - e)$.

Method 3:

Let $u = 2x - y$ ; $v = \frac{x}{y}$, then we can get $|J| = |\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right|| = \dfrac{u}{(2v-1)^2}$

Therefore, $\iint_R e^{\frac{x+2y}{2x-y}} dA = \int_{\frac{4}{3}}^3 \int_1^2 e^{\frac{v+2}{2v-1}} |J| du dv = \int_{\frac{4}{3}}^3 \int_1^2 e^{\frac{v+2}{2v-1}} \frac{u}{(2v-1)^2} du dv$

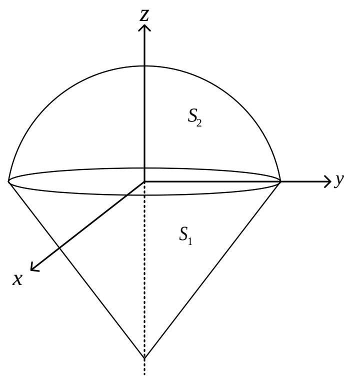
$= \frac{3}{10}(e^2 - e)$.

**給分標準: $J$ 算錯扣4分 積分上下界找錯扣4分 前面兩項找對答案算錯扣2分.**

5. (16%) Let $E$ be the space region bounded by the surfaces

$$S_1 := \{(x,y,z) |\ z = -1 + \sqrt{x^2 + y^2},\ z \le 0\},$$
$$S_2 := \{(x,y,z) |\ z = 1 - x^2 - y^2,\ z \ge 0\},$$

and $\mathbf{V}(x,y,z) = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$.




(a) (6%) Evaluate $\iiint_\Omega \text{div}\mathbf{V}\, dV$, where $\Omega$ is the solid region enclosed by $S_1 \cup S_2$.

(b) (4%) State the Divergence Theorem and evaluate the total outward flux $\iint_{S_1 \cup S_2} \mathbf{V} \cdot \mathbf{dS}$.

(c) (6%) Compute the upward flux of $\mathbf{V}$ across $S_2$.

**Solution:**

**Solution 5(a).**

$$\iiint_\Omega \text{div} V\, dV = \iiint_\Omega 1 dV$$
$$= \int_0^{2\pi} \int_0^1 \int_{-1+r}^{1-r^2} r dz dr d\theta$$
$$= \frac{5\pi}{6}$$

Note:

1. The shape of region 2 is not hemisphere.
2. div$V = 1$, 2points
3. Volume of region 1 $= \frac{\pi}{3}$, 2points

4. Volume of region 2 $=\frac{\pi}{2}$, 2points

**Solution 5(b).**
Divergence Theorem:
$\Omega$: simple solid region
$S$:boudary surface of $\Omega$ with positive outward orientation.
$V = (P, Q, R)$: vector field with $P_x, Q_y, R_z$ continuous on an open region that contains $\Omega$.
Then we have

$$\iiint_{\Omega} V dV = \iint_{S} V \cdot dS$$
$$\therefore \iint_{S_1 \cup S_2} V \cdot dS = \frac{5\pi}{6}$$

Note:

1. $\Omega$, 1point
2. $P_x, Q_y, R_z$, 1point
3. $\iiint_{\Omega} V dV = \iint_{S} V \cdot dS$, 1point
4. $\iint_{S_1 \cup S_2} V \cdot dS = \frac{5\pi}{6}$, 1point

**Solution 5(c).**
**Method 1: Direct calculation**

$$\begin{aligned} r(u,v) &= (u\cos v, u\sin v, 1-u^2), \quad 0 \le u \le 1, 0 \le v \le 2\pi \\ r_u &= (\cos v, \sin v, -2u) \\ r_v &= (-u\sin v, u\cos v, 0) \\ r_u \times r_v &= (2u^2\cos v, 2u^2 \sin v, u) \\ V \cdot r_u \times r_v &= u - u^3 \end{aligned}$$
$$\therefore \iint_{S_2} V \cdot dS = \int_0^{2\pi} \int_0^1 u - u^3 du dv = \frac{\pi}{2}$$

**Method 2**
Let $S_3 = \{(x,y,z) | x^2 + y^2 \le 1, z = 0\}$. Then $V \cdot n = 0$ on $S_3$ where $n = -k$ which implies

$$\iint_{S_3} V \cdot dS = 0$$

This leads to

$$\therefore \iint_{S_2} V \cdot dS = \iiint_{\Omega_2} V dV - \iint_{S_3} V \cdot dS = \frac{\pi}{2}$$

where volume of region 2 has been calculated in part $(a)$.
Note: $\iint_{S_2} V \cdot dS = \iiint_{\Omega_2} V dV$ is WRONG.

**Method 3**

$$\begin{aligned} \iint_{S_2} V \cdot dS &= \iiint_{\Omega} V dV - \iint_{S_1} V \cdot dS \\ &= \frac{5\pi}{6} - \frac{\pi}{3} \\ &= \frac{\pi}{2} \end{aligned}$$

where the surface integral of $S_1$ is as follows:

$$\begin{aligned} r(u,v) &= (u\cos v, u\sin v, -1+u), \quad 0 \le u \le 1, 0 \le v \le 2\pi \\ r_u &= (\cos v, \sin v, 1) \\ r_v &= (-u\sin v, u\cos v, 0) \\ r_v \times r_u &= (u\cos v, u\sin v, -u) \\ V \cdot r_u \times r_v &= u - u^2 \end{aligned}$$

$$\therefore \iint_{S_2} V \cdot dS = \int_0^{2\pi} \int_0^1 u - u^2 du dv = \frac{\pi}{3}$$

Note: Be careful of the sign of the normal

6. (12%) Evaluate $\oint_C (x^2y^2 + y)dx - (2xy^3 - 3x)dy$, where $C$ is described by the polar equation $r = 1 - \cos\theta$ oriented counterclockwise.

**Solution:**

Let $D = \{0 \le r \le 1 - \cos\theta, 0 \le \theta \le 2\pi\}$ be the region bound by the curve $C$. Apply Green's Theorem,

$$\oint_C (x^2y^2 + y)\,dx - (2xy^3 - 3x)\,dy = \iint_D (-2y^3 + 3) - (2x^2y + 1)\,dA$$
$$= \iint_D (2 - 2x^2y - 2y^3)\,dA. \qquad \text{(6 points)}$$

Use polar coordinate $x = r\cos\theta$, $y = r\sin\theta$,

$$\iint_D (2 - 2x^2y - 2y^3)\,dA = \int_0^{2\pi} (2 - 2r^3\sin\theta)\,r\,dr\,d\theta \qquad \text{(5 points)}$$
$$= \int_0^{2\pi} \left( r^2 - \frac{2}{5}r^5\sin\theta \Big|_{r=0}^{r=1-\cos\theta} \right) d\theta$$
$$= \int_0^{2\pi} (1 - \cos\theta)^2 - \frac{2}{5}\sin\theta(1 - \cos\theta)^5\,d\theta$$
$$= 2\pi + 0 + \frac{1}{2}\cdot 2\pi - \left[\frac{2}{30}(1 - \cos\theta)^6\right]_{\theta=0}^{\theta=2\pi} = 3\pi. \qquad \text{(1 points)}$$

7. (12%) Find the area of the sphere $x^2 + y^2 + z^2 = 4$ lying inside the cylinder $(x - 1)^2 + y^2 = 1$.

**Solution:**

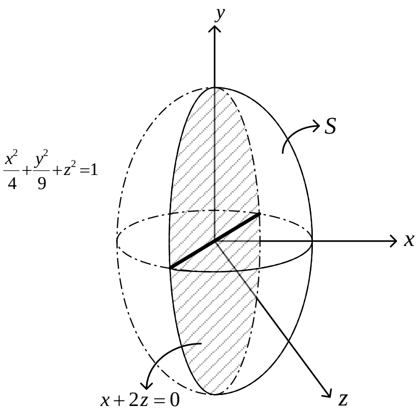
The area

$$= 2\int\int_{(x-1)^2+y^2\le 1} \sqrt{1 + z_x^2 + z_y^2}\,dA \qquad \text{[3 points]}$$
$$(z = \sqrt{4 - x^2 - y^2}, z_x = \frac{-x}{\sqrt{4 - x^2 - y^2}}, z_y = \frac{-y}{\sqrt{4 - x^2 - y^2}}) \qquad \text{[1 points]}$$
$$= 2\int\int_{(x-1)^2+y^2\le 1} \frac{2}{\sqrt{4 - x^2 - y^2}}\,dA \qquad \text{[1 points]}$$

(Use the polar coordinate to calculate)

$$= 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{2}{\sqrt{4 - r^2}} r dr d\theta \qquad \text{[3 points]}$$
$$= 4\int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} (4 - r^2)^{-\frac{1}{2}} dr^2 d\theta \qquad \text{[1 points]}$$
$$= 8\int_0^{\frac{\pi}{2}} -(4 - r^2)^{\frac{1}{2}} |_{r=0}^{r=2\cos\theta} d\theta \qquad \text{[1 points]}$$
$$= 8\int_0^{\frac{\pi}{2}} 2 - 2\sin\theta d\theta \qquad \text{[1 points]}$$
$$= 8(\pi - 2) \qquad \text{[1 points]}$$

8. (14%) Compute $\iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S}$, where $\mathbf{F} = e^{xz}\mathbf{i} + (x^2 + z^2)\mathbf{j} + (y + \cos z)\mathbf{k}$ and where $S = \left\{(x, y, z)\Big| \frac{x^2}{4} + \frac{y^2}{9} + z^2 = 1 \text{ and } x + 2z \ge 0\right\}$ oriented so that the boundary is counterclockwise when viewed from above.




**Solution:**

**Method 1:**
$curlF = \nabla \times F = \langle 1-2z, xe^{xz}, 2x\rangle$ (3 pts)

Let $T$ be the intersection of the ellipsoid with the plane $x+2z=0$. Note that the boundary of $S$ and $T$ are the same. On the plane $x+2z=0$ which is $z=-\frac{1}{2}x$.
Parametrize $T$ by $\gamma(x,y)=(x,y,\frac{-x}{2})$ with the condition $\frac{x^2}{2}+\frac{y^2}{9}\le 1$ ($\frac{x^2}{4}+\frac{y^2}{9}+z^2\le 1$).
$\gamma_x=(1,0,\frac{-1}{2}), \gamma_y=(\frac{1}{2},0,1) \Rightarrow \gamma_x\times\gamma_y=(\frac{1}{2},0,1)$ (3 pts)
Let $D$ be the projection of $T$ onto $x-y$ plane.
Now, by Stoke's Theorem

$$\begin{aligned}\int\int_S curlF\cdot dS &= \int\int_T curlF\cdot dS\\ &= \int\int_D \langle 1+x, xe^{-\frac{1}{2}x^2}, 2x\rangle\cdot(\frac{1}{2},0,1)dxdy\\ &= \int\int_{\frac{x^2}{2}+\frac{y^2}{9}\le 1}\frac{1}{2}+\frac{5}{2}xdxdy \quad \text{(6 pts)}\end{aligned}$$

By symmetry, $\int\int_D \frac{5}{2}xdxdy=0$.
So

$$\int\int_{\frac{x^2}{2}+\frac{y^2}{9}\le 1}\frac{1}{2}+\frac{5}{2}xdxdy=\frac{1}{2}Area(D)=\frac{1}{2}\cdot\pi\cdot\sqrt{2}\cdot 3=\frac{3\sqrt{2}}{2}\pi \quad \text{(2 pts)}$$

Thus $\int\int_S curlF\cdot dS=\frac{3\sqrt{2}}{2}\pi$.

**Method 2:**
$C$ is the intersection of $\frac{x^2}{4}+\frac{y^2}{9}+z^2=1$ and $x+2z=0$.
We have $z=-\frac{x}{2}$ and $\frac{x^2}{2}+\frac{y^2}{9}=1$. So we can parametrize $C$ as

$$\gamma(\theta)=(\sqrt{2}\cos(\theta), 3\sin(\theta), -\frac{\sqrt{2}\cos(\theta)}{2}), 0\le\theta\le 2\pi \text{ (3 pts)}$$

On $C$ , we have

$$\begin{aligned}F &= \langle e^{-\cos^2(\theta)}, \frac{5}{2}\cos^2(\theta), 3\sin(\theta)+\cos(-\frac{\sqrt{2}\cos(\theta)}{2})\rangle \quad \text{(1 pt)}\\ \gamma'(\theta) &= \langle -\sqrt{2}\sin(\theta), 3\cos(\theta), \frac{\sqrt{2}\sin(\theta)}{2}\rangle \quad \text{(1 pt)}\\ \Rightarrow F\cdot\gamma'(\theta) &= -\sqrt{2}\sin(\theta)e^{-\cos^2(\theta)}+\frac{15}{2}\cos^3(\theta)+\frac{3\sqrt{2}\sin^2(\theta)}{2}+\frac{\sqrt{2}\sin(\theta)\cos(-\frac{\sqrt{2}\cos(\theta)}{2})}{2}\end{aligned}$$

Then

$$\int F\cdot dr=\int_0^{2\pi}\Big(-\sqrt{2}\sin(\theta)e^{-\cos^2(\theta)}+\frac{15}{2}\cos^3(\theta)+\frac{3\sqrt{2}\sin^2(\theta)}{2}+\frac{\sqrt{2}\sin(\theta)\cos(-\frac{\sqrt{2}\cos(\theta)}{2})}{2}\Big)d\theta \text{ (3 pts)}$$

Note that $f(\theta) = -\sqrt{2}\sin(\theta)e^{-\cos^2(\theta)} + \frac{\sqrt{2}\sin(\theta)\cos(-\frac{\sqrt{2}\cos(\theta)}{2})}{2}$ is an odd function.
Thus $\int_0^{2\pi} f(\theta)d\theta = \int_{-\pi}^{\pi} f(\theta)d\theta = 0$.
We also have $\int_0^{2\pi} \cos^3(\theta)d\theta = 0$.

So $\int_C F \cdot dr = \int_0^{2\pi} \frac{3\sqrt{2}\sin^2(\theta)}{2}d\theta = \int_0^{2\pi} \frac{3\sqrt{2}(1-\cos(2\theta))}{4} = \frac{3\sqrt{2}\cdot 2\pi}{4} = \frac{3\sqrt{2}\pi}{2}$ (3 pts)

Thus, by Stoke's Theorem $\int\int_S curlF \cdot dS = \frac{3\sqrt{2}\pi}{2}$ (3 pts).