1052微甲01-04班期末考解答和評分標準

1. (12%) Determine whether the statement is true of false. Fill T (true) or F (false) in the blanks. If the statement is false, write down a reason, or give a correct statement, or find a counterexample.

   (a) Let $\mathbf{F}(x, y) = P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}$. If $\dfrac{\partial Q}{\partial x} = \dfrac{\partial P}{\partial y}$ throughout the domain of $\mathbf{F}(x, y)$, then the line integrals of $\mathbf{F}(x, y)$ is independent of path on the domain.

   (b) Let $f(x, y)$ be a smooth function. Suppose that a smooth curve C gives an orientation from initial point p to terminal point q. If $-C$ denotes the curve consisting of the same points as C but with the opposite orientation (from initial point q to terminal point p), then $\displaystyle\int_{-C} f(x, y)\,\mathrm{d}s = -\int_C f(x, y)\,\mathrm{d}s$.

   (c) For a unit circle $C : x^2 + y^2 = 1$, we have $\displaystyle\oint_C x\,\mathrm{d}y = 0$ by symmetry.

   (d) Any smooth function $f(x, y, z)$ satisfies $\mathrm{div}(\nabla f) = 0$.

Solution:

(a) F (1 point)
The domain of $\mathbf{F}(x, y)$ should be open simply-connected (2 points)

(b) F (1 point)
$\displaystyle\int_{-C} f(x, y)\mathrm{d}s = \int_C f(x, y)\mathrm{d}s$ (2 points)

(c) F (1 point)
$\displaystyle\oint_C x\mathrm{d}y$ =the area of the unit circle= π (2 points)

(d) F (1 point)
A counterexample by $f(x, y, z) = x^2$, then $\mathrm{div}(\nabla f) = 2$ (2 points)

2. (12%) Evaluate the following integrals.

   (a) $\displaystyle\int_0^1\int_y^1 \tan(x^2)\,\mathrm{d}x\,\mathrm{d}y$.

   (b) $\displaystyle\int_{\frac{1}{\sqrt{2}}}^1\int_{\sqrt{1-x^2}}^x 1\,\mathrm{d}y\,\mathrm{d}x + \int_1^{\sqrt{2}}\int_0^x 1\,\mathrm{d}y\,\mathrm{d}x + \int_{\sqrt{2}}^2\int_0^{\sqrt{4-x^2}} 1\,\mathrm{d}y\,\mathrm{d}x$.

Solution:

(a) $\displaystyle\int \tan\left(x^2\right)dx$ is not an elementary function, so we must change the order of integration.

$$\begin{aligned}
&\int_0^1\int_y^1 \tan\left(x^2\right)dxdy \\
=\ &\int_0^1\int_0^x \tan\left(x^2\right)dydx \quad \textbf{(3pts)} \\
=\ &\int_0^1 x\tan\left(x^2\right)dx \quad \textbf{(1pt)} \\
=\ &\frac{1}{2}\ln\left|\sec\left(x^2\right)\right|\Big|_0^1 = \frac{1}{2}\ln\left(\sec(1)\right) \quad \textbf{(2pts)}
\end{aligned}$$

(b) The region is easily described using polar coordinates.

$$\begin{aligned}
&\int_{\frac{1}{\sqrt{2}}}^{1}\int_{\sqrt{1-x^2}}^{x} 1dydx + \int_{1}^{\sqrt{2}}\int_{0}^{x} 1dydx + \int_{\sqrt{2}}^{2}\int_{0}^{\sqrt{4-x^2}} 1dydx \\
= &\int_{0}^{\frac{\pi}{4}}\int_{1}^{2} rdrd\theta \quad \textbf{(4pts)} \\
= &\int_{1}^{2} rdr \int_{0}^{\frac{\pi}{4}} d\theta \\
= &\frac{1}{2}r^2\Big|_1^2 \cdot \theta\Big|_0^{\frac{\pi}{4}} \\
= &\frac{3}{8}\pi \quad \textbf{(2pts)}
\end{aligned}$$

3. (12%) Evaluate the following integrals.

(a) $\iiint_E y\cos((y-z)^2)\,\mathrm{d}V$, where E is the solid tetrahedron bounded by four planes $x = 1, y = 1, z = 0$, and $x + y - z = 1$, as Figure 1.

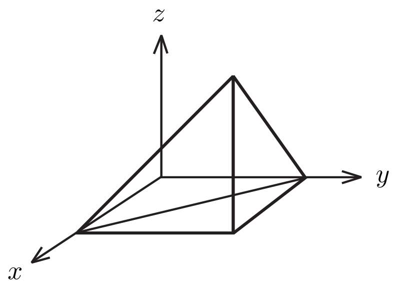

Figure 1: The tetrahedron.

(b) $\displaystyle\int_0^{\frac{1}{\sqrt{2}}}\int_x^{\sqrt{1-x^2}}\int_{2\sqrt{x^2+y^2}}^{1+x^2+y^2} 1\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x.$

Solution:

(a). $\displaystyle\int_0^1\int_0^y\int_{1-y+z}^{1} y\cos\left((y-z)^2\right) dxdzdy$ (4%, including the correct order)

$\displaystyle= \int_0^1\int_0^y (y-z)y\cos\left((y-z)^2\right) dzdy$

$\displaystyle= \int_0^1 \frac{1}{2}y\sin\left(y^2\right) dy$

$\displaystyle= \frac{1}{4} - \frac{1}{4}\cos 1$ (2%)

(b). $\displaystyle\int_{\pi/4}^{\pi/2}\int_0^1\int_{2r}^{1+r^2} 1\cdot rdzdrd\theta$ (4%)

$\displaystyle= \frac{\pi}{4}\int_0^1 r\left(1+r^2-2r\right) dr$

$\displaystyle= \frac{\pi}{48}$ (2%)

4. (10%) Evaluate the double integral $\iint_R e^{3x^2+y^2}\,dA$, where R is the region inside the ellipse $3x^2+y^2=1$ and above the lines $y=x$ and $y=-\sqrt{3}\,x$, as Figure 2.

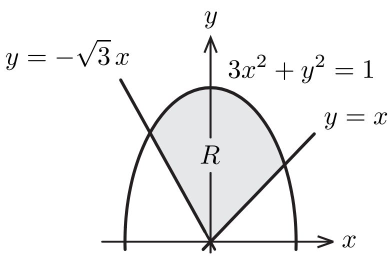

Figure 2: The region R.

Solution:

Let $x = \frac{1}{\sqrt{3}} \cdot r\cos\theta, y = r\sin\theta$

$\iint_R e^{3x^2+y^2}\,dxdy$

$= \int_{\pi/6}^{3\pi/4}\int_0^1 e^{r^2}\cdot\frac{1}{\sqrt{3}}\cdot r dr d\theta$ (2% for deformation, 2% for polar coordinate, 1% for Jacobian determinant, and 3% for correct intervals)

$= \frac{1}{\sqrt{3}}\cdot\frac{7}{24}\pi(e-1)(2\%)$

5. (12%) Evaluate the surface integral $\iint_S \sqrt{x^2+y^2}\,dS$, where S is the part of the surface $z=\tan^{-1}\left(\frac{y}{x}\right)$ inside the circular cylinder $x^2+y^2=1$ and in the first octant.

Solution:

Step1.

We can write the parametric equations of S as

$$x = r\cos\theta \quad y = r\sin\theta \quad z=\theta$$

where the parameter domain is

$$D = \{(r,\theta)\big|0 < r \le 1, 0 \le \theta \le \frac{\pi}{2}\}$$

and the vector equation is

$$\mathbf{r}(r,\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j} + \theta\mathbf{k}$$

(**3pts**)

Step2.

Find $|\mathbf{r}_r \times \mathbf{r}_\theta|$.

$$\begin{cases} \mathbf{r}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j} + 0\mathbf{k} \\ \mathbf{r}_\theta = -r\sin\theta\mathbf{i} + r\cos\theta\mathbf{j} + 1\mathbf{k} \end{cases}$$

$$\Rightarrow \quad \mathbf{r}_r \times \mathbf{r}_\theta = \sin\theta\mathbf{i} + (-\cos\theta)\mathbf{j} + r\mathbf{k}$$

$$\Rightarrow \quad |\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{1+r^2} \quad (\mathbf{3pts})$$

Step3.

Evaluate $\iint_S \sqrt{x^2+y^2}dS$.

$$
\begin{aligned}
\iint_S \sqrt{x^2+y^2}dS &= \iint_D \sqrt{(r\cos\theta)^2+(r\sin\theta)^2}\cdot|\mathbf{r}_r\times\mathbf{r}_\theta|drd\theta \quad (\mathbf{3pts})\\
&= \int_0^{\frac{\pi}{2}}\int_0^1 r\sqrt{1+r^2}drd\theta\\
&= \int_0^1 r\sqrt{1+r^2}dr\int_0^{\frac{\pi}{2}}d\theta\\
&= \frac{1}{3}(1+r^2)^{\frac{3}{2}}\Big|_0^1\cdot\theta\Big|_0^{\frac{\pi}{2}}\\
&= \frac{\pi}{6}(2^{\frac{3}{2}}-1) \quad (\mathbf{3pts})
\end{aligned}
$$

6. (12%) Let $\mathbf{F}(x,y)=\dfrac{x+y}{x^2+y^2}\mathbf{i}+\dfrac{-x+y}{x^2+y^2}\mathbf{j}$.

(a) Is $\mathbf{F}(x,y)$ conservative on the half plane $D=\{(x,y)|x>0\}$?

(b) Evaluate the line integral $\int_{C_1}\mathbf{F}\cdot d\mathbf{r}$, where C_1 is the part of the parabola $y=(x-2)^2$ from $(2,0)$ to $(4,4)$.

(c) Evaluate the line integral $\int_{C_2}\mathbf{F}\cdot d\mathbf{r}$, where C_2 is the unit circle $x^2+y^2=1$ oriented counterclockwise. Is $\mathbf{F}(x,y)$ conservative on $\mathbb{R}^2-\{(0,0)\}$?

Solution:

$$\mathbf{F}=\left(\frac{x+y}{x^2+y^2},\frac{-x+y}{x^2+y^2}\right)$$

(a)
(Method 1)

$$
\begin{aligned}
&\int\frac{-x+y}{x^2+y^2}\,dy=\frac{1}{2}\ln(x^2+y^2)-\tan^{-1}(\frac{y}{x})+C(x)\\
&\text{Let }\frac{\partial}{\partial x}\left[\frac{1}{2}\ln(x^2+y^2)-\tan^{-1}(\frac{y}{x})+C(x)\right]=\frac{x+y}{x^2+y^2}\\
&\Longrightarrow \frac{x}{x^2+y^2}-\frac{\frac{-y}{x^2}}{1+(\frac{y}{x})^2}+C'(x)=\frac{x+y}{x^2+y^2}\\
&\Longrightarrow C'(x)=0\\
&\Longrightarrow C(x)=K\text{ is a constant.}
\end{aligned}
$$

We already find a function

$$\phi(x,y)=\frac{1}{2}\ln(x^2+y^2)-\tan^{-1}(\frac{y}{x})+K \qquad (3\%)$$

such that

$$\mathbf{F}=\nabla\phi$$

for all (x,y) in D. Therefore, $\mathbf{F}$ is conservative on D. (1%)

(Method 2)
Let

$$P(x,y)=\frac{x+y}{x^2+y^2}\text{ and }Q(x,y)=\frac{-x+y}{x^2+y^2}$$

Since D is an open simply-connected region, P and Q have continuous first-order partial derivatives, and

$$\frac{\partial P}{\partial y}=\frac{x^2-2xy-y^2}{(x^2+y^2)^2}=\frac{\partial Q}{\partial x} \qquad (3\%)$$

through out D, we conclude that $\mathbf{F}$ is conservative on D. (1%)

(b)
(Method 1) Since the path C_1 lies in D, we have

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \phi(4,4)-\phi(2,0) \qquad (3\%)\\ &= \frac{3}{2}\ln 2 - \frac{\pi}{4} \qquad (1\%)\end{aligned}$$

(Method 2) Because $\mathbf{F}$ is conservative on D, $\int_C \mathbf{F}\cdot d\mathbf{r}$ is independent of path on D. We can choose paths $\alpha : 2 \le x \le 4, y = 0$, and $\beta : x = 4, 0 \le y \le 4$ such that the path $\alpha \cup \beta$ goes from $(2,0)$ to $(4,0)$ horizontally and then goes from $(4,0)$ to $(4,4)$ vertically. Therefore,

$$\begin{aligned}\int_{C_1} \mathbf{F}\cdot d\mathbf{r} &= \int_{\alpha} \mathbf{F}\cdot d\mathbf{r} + \int_{\beta} \mathbf{F}\cdot d\mathbf{r}\\ &= \int_{x=2}^{4} \frac{1}{x}\, dx + \int_{y=0}^{4} \frac{-4+y}{16+y^2}\, dy \qquad (2\%)\\ &= \ln|x|\Big|_2^4 - \tan^{-1}(\frac{y}{4})\Big|_0^4 + \frac{1}{2}\ln(16+y^2)\Big|_0^4 \qquad (1\%)\\ &= \frac{3}{2}\ln 2 - \frac{\pi}{4} \qquad (1\%)\end{aligned}$$

(c) We can parametrize $C_2 : \mathbf{r}(\theta) = (\cos\theta, \sin\theta), 0 \le \theta \le 2\pi$, and the vector field becomes $\mathbf{F}(\cos\theta,\sin\theta) = (\cos\theta+\sin\theta, -\cos\theta+\sin\theta)$ on C_2. (1%)
Then,

$$\begin{aligned}\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} &= \int_{\theta=0}^{2\pi} (\cos\theta+\sin\theta, -\cos\theta+\sin\theta)\cdot(-\sin\theta,\cos\theta)d\theta\\ &= \int_{\theta=0}^{2\pi} -\sin^2\theta - \cos^2\theta d\theta = -2\pi \qquad (2\%)\end{aligned}$$

Remark: You can also observe that the normal vector (x, y) of the circle is perpendicular to (dx, dy), and then simplify

$$\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_2} \left(\frac{y}{x^2+y^2}, \frac{-x}{x^2+y^2}\right)\cdot(dx,dy) = \int_{\theta=0}^{2\pi} -\sin^2\theta - \cos^2\theta d\theta = -2\pi.$$

Note that C_2 is a closed path on $\mathbb{R}^2\backslash\{(0,0)\}$ but $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} \neq 0$, which implies that $\int_C \mathbf{F}\cdot d\mathbf{r}$ is not independent of path on $\mathbb{R}^2\backslash\{(0,0)\}$. Therefore, $\mathbf{F}$ is not conservative on $\mathbb{R}^2\backslash\{(0,0)\}$. (1%)

7. (12%) Find the value $k \in \mathbb{R}$ such that the line integral

$$I(k) = \int_{C_k} (1 + y^2 + y\,\mathrm{e}^{xy})\,\mathrm{d}x + (2x + y + x\,\mathrm{e}^{xy})\,\mathrm{d}y$$

achieves the minimum value, where C_k is the curve $y = k\sin x$ from $(0,0)$ to $(\pi, 0)$.

Solution:
Let $\mathbf{F}(x,y) = (1+y^2+ye^{xy})\,\mathbf{i} + (2x+y+xe^{xy})\,\mathbf{j}$
and $C_1 : \mathbf{r}(t) = t\mathbf{i}$ where t from π to 0
$P(x,y) = 1+y^2+ye^{xy}$, $Q(x,y) = 2x+y+xe^{xy}$
$\Rightarrow I(k) + \int_{C_1} \mathbf{F}\cdot\mathrm{d}\mathbf{r} = \iint_D (\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})\mathrm{d}A$ (5 points)
To achieve $I(k)$ minimum value is to achieve $\iint_D (\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x})\mathrm{d}A$ minimum value

$$\begin{aligned}
\iint_D \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) dA &= \iint_D (2y-2)\,dA \\
&= \int_0^{\pi}\int_0^{k\sin x} (2y-2)\,dy\,dx \\
&= \int_0^{\pi} (k^2\sin^2 x - 2k\sin x)\,dx \\
&= \frac{\pi}{2}k^2 - 4k \\
&= \frac{\pi}{2}\left(k - \frac{4}{\pi}\right)^2 - \frac{8}{\pi} \text{ (6 points)}
\end{aligned}$$

$\therefore k = \dfrac{4}{\pi}$ (1 point)

8. (13%) Compute the line integral

$$\int_C z^2\,\mathrm{d}x - x^2\,\mathrm{d}y + 2yz\,\mathrm{d}z,$$

where C is the curve of intersection of the upper half sphere $z = \sqrt{4 - x^2 - y^2}$ and the circular cylinder $x^2 + y^2 = 2y$, orientated counterclockwise viewed from the above, as Figure 3.

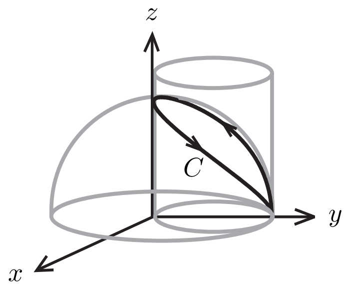

Figure 3: The curve C.

Solution:

Method I Straightforward Computation with $\mathbf{F} = \left(z^2, -x^2, 2yz\right)$

Solution 1 Parameterize C by $\mathbf{r}(t) = \left(\cos t, 1 + \sin t, \sqrt{2 - 2\sin t}\right)$, t from 0 to 2π.

$$\text{Then}\quad \mathbf{F} = \left(2 - 2\sin t, -\cos^2 t, 2(1+\sin t)\sqrt{2 - 2\sin t}\right)$$

$$\mathbf{r}'(t) = \left(-\sin t, \cos t, \frac{-\cos t}{\sqrt{2 - 2\sin t}}\right)$$

$$\begin{aligned}
\text{So we compute } \int_C \mathbf{F}\cdot d\mathbf{r} &= \int_0^{2\pi} -2\sin t + 2\sin^2 t - \cos^3 t - 2\cos t - 2\sin t\cos t\,dt \\
&= \int_0^{2\pi} 2\sin^2 t\,dt = \int_0^{2\pi} 1 - \cos 2t\,dt = 2\pi.
\end{aligned}$$

Solution 2 Parameterize C by $\mathbf{r}(t) = \left(\sin t, 1 + \cos t, 2\sin\frac{t}{2}\right)$, t from $\mathbf{2\pi}$ to $\mathbf{0}$.

$$\text{Then}\quad \mathbf{F} = \left(4\sin^2\frac{t}{2}, -\sin^2 t, 4(1+\cos t)\sin\frac{t}{2}\right)$$

$$\mathbf{r}'(t) = \left(\cos t, -\sin t, \cos\frac{t}{2}\right)$$

$$\text{So we compute } \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{2\pi}^{0} 4\cos t\sin^2\frac{t}{2} + \sin^3 t + 4(1+\cos t)\sin\frac{t}{2}\cos\frac{t}{2}\,dt$$
$$= \int_{2\pi}^{0} 4\cos t\sin^2\frac{t}{2}\,dt = \int_{2\pi}^{0} 2\cos t(1-\cos t)\,dt$$
$$= \int_{0}^{2\pi} 2\cos^2 t\,dt = \int_0^{2\pi} 1+\cos 2t\,dt = 2\pi.$$

Solution 3 Parameterize C by $\mathbf{r}(t) = \big(\sin 2t, 2\sin^2 t, 2|\cos t|\big)$, t from 0 to π.

$$\text{Then}\quad \mathbf{F} = \big(4\cos^2 t, -\sin^2 2t, 8\sin^2 t|\cos t|\big)$$
$$\mathbf{r}'(t) = \left(2\cos 2t, 2\sin 2t, \frac{-\sin 2t}{|\cos t|}\right)$$

$$\text{So we compute } \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{\pi} 8\cos 2t\cos^2 t - 2\sin^3 2t - 8\sin^2 t\sin 2t\,dt$$
$$= \int_0^{\pi} 8\cos 2t\cos^2 t\,dt = \int_0^{\pi} 4\cos 2t(1+\cos 2t)\,dt$$
$$= \int_0^{\pi} 4\cos 2t + 4\cos^2 2t\,dt = \int_0^{\pi} 4\cos^2 2t\,dt$$
$$= \int_0^{\pi} 2+2\cos 4t\,dt = 2\pi.$$

Method II Applying the Stokes Theorem with $\mathbf{F} = \big(z^2, -x^2, 2yz\big)$, $\text{curl}\mathbf{F} = (2z, 2z, -2x)$

Solution 1 Let S be the portion of the sphere with boundary C. By Stokes' Theorem, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_D \text{curl}\mathbf{F}\cdot\mathbf{n}\ \frac{1}{|\mathbf{n}\cdot\mathbf{k}|}\,dxdy$$
$$= \iint_D \left[(2z,2z,-2x)\cdot\left(\frac{x}{2},\frac{y}{2},\frac{z}{2}\right)\right]\frac{2}{z}\,dxdy, \quad D: x^2+(y-1)^2\le 1$$
$$= \iint_D 2y\,dxdy = \int_0^{2\pi}\int_0^1 2(1+r\sin\theta)r\,drd\theta = \int_0^{2\pi}\int_0^1 2r\,drd\theta = 2\pi.$$

Solution 2 Let S be the portion of the sphere with boundary C. Parameterize S by $\mathbf{r}(x,y)$ where

$$\mathbf{r}(x,y) = \left(x, y, \sqrt{4-x^2-y^2}\right), \quad x^2+(y-1)^2\le 1$$
$$\mathbf{r}_x(x,y) = \left(1, 0, \frac{-x}{\sqrt{4-x^2-y^2}}\right)$$
$$\mathbf{r}_y(x,y) = \left(0, 1, \frac{-y}{\sqrt{4-x^2-y^2}}\right)$$
$$\mathbf{r}_x\times\mathbf{r}_y = \left(\frac{x}{\sqrt{4-x^2-y^2}}, \frac{y}{\sqrt{4-x^2-y^2}}, 1\right), \text{ pointing upward}$$
$$\text{curl}\mathbf{F} = \left(2\sqrt{4-x^2-y^2}, 2\sqrt{4-x^2-y^2}, -2x\right)$$

By Stokes' Theorem, we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \text{curl}\mathbf{F}\cdot\mathbf{r}_x\times\mathbf{r}_y\,dxdy$$
$$= \iint_{x^2+(y-1)^2\le 1} 2y\,dxdy = \int_0^{2\pi}\int_0^1 2(1+r\sin\theta)r\,drd\theta = \int_0^{2\pi}\int_0^1 2r\,drd\theta = 2\pi.$$

Solution 3 Let S be the portion of the sphere with boundary C. Parameterize S by $\mathbf{r}(u,v)$ where

$$\begin{aligned}
\mathbf{r}(u,v) &= \left(u\cos v, 1+u\sin v, \sqrt{3-u^2-2u\sin v}\right), \quad 0\le u\le 1, 0\le v\le 2\pi\\
\mathbf{r}_u(u,v) &= \left(\cos v, \sin v, \frac{-u-\sin v}{\sqrt{3-u^2-2u\sin v}}\right)\\
\mathbf{r}_v(u,v) &= \left(-u\sin v, u\cos v, \frac{-u\cos v}{\sqrt{3-u^2-2u\sin v}}\right)\\
\mathbf{r}_u\times\mathbf{r}_v &= \left(\frac{u^2\cos v}{\sqrt{3-u^2-2u\sin v}}, \frac{u+u^2\sin v}{\sqrt{3-u^2-2u\sin v}}, u\right), \text{ pointing upward}\\
\text{curl}\mathbf{F} &= \left(2\sqrt{3-u^2-2u\sin v}, 2\sqrt{3-u^2-2u\sin v}, -2u\cos v\right)
\end{aligned}$$

By Stokes' Theorem, we have

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \text{curl}\mathbf{F}\cdot\mathbf{r}_u\times\mathbf{r}_v\,dudv\\
&= \int_0^{2\pi}\int_0^1 2u+2u^2\sin v\,dudv = \int_0^{2\pi}\int_0^1 2u\,dudv = 2\pi.
\end{aligned}$$

Solution 4 Let S be the portion of the sphere with boundary C. Parameterize S by $\mathbf{r}(u,v)$ where

$$\begin{aligned}
\mathbf{r}(u,v) &= \left(u\cos v, u\sin v, \sqrt{4-u^2}\right), \quad 0\le u\le 2\sin v, 0\le v\le \pi\\
\mathbf{r}_u(u,v) &= \left(\cos v, \sin v, -\frac{u}{\sqrt{4-u^2}}\right)\\
\mathbf{r}_v(u,v) &= (-u\sin v, u\cos v, 0)\\
\mathbf{r}_u\times\mathbf{r}_v &= \left(\frac{u^2\cos v}{\sqrt{4-u^2}}, \frac{u^2\sin v}{\sqrt{4-u^2}}, u\right), \text{ pointing upward}\\
\text{curl}\mathbf{F} &= \left(2\sqrt{4-u^2}, 2\sqrt{4-u^2}, -2u\cos v\right)
\end{aligned}$$

By Stokes' Theorem, we have

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \text{curl}\mathbf{F}\cdot\mathbf{r}_u\times\mathbf{r}_v\,dudv\\
&= \int_0^{\pi}\int_0^{2\sin v} 2u^2\sin v\,dudv = \frac{16}{3}\int_0^{\pi}\sin^4 v\,dv\\
&= \frac{16}{3}\int_0^{\pi}\frac{1}{8}\left(2-4\cos 2v+1-\cos 4v\right)\,dv = 2\pi.
\end{aligned}$$

Solution 5 Let S be the portion of the sphere with boundary C. Parameterize S by $\mathbf{r}(u,v)$ where

$$\begin{aligned}
\mathbf{r}(u,v) &= (2\sin u\cos v, 2\sin u\sin v, 2\cos u), \quad 0\le u\le\frac{\pi}{2},\ u\le v\le \pi-u.\\
\mathbf{r}_u(u,v) &= (2\cos u\cos v, 2\cos u\sin v, -2\sin u)\\
\mathbf{r}_v(u,v) &= (-2\sin u\sin v, 2\sin u\cos v, 0)\\
\mathbf{r}_u\times\mathbf{r}_v &= \left(4\sin^2 u\cos v, 4\sin^2 u\sin v, 4\sin u\cos u\right), \text{ pointing upward}\\
\text{curl}\mathbf{F} &= (4\cos u, 4\cos u, -4\sin u\cos v)
\end{aligned}$$

By Stokes' Theorem, we have

$$\begin{aligned}
\int_C \mathbf{F}\cdot d\mathbf{r} &= \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \text{curl}\mathbf{F}\cdot\mathbf{r}_u\times\mathbf{r}_v\,dudv\\
&= \int_0^{\frac{\pi}{2}}\int_u^{\pi-u} 16\sin^2 u\cos u\sin v\,dvdu = \int_0^{\frac{\pi}{2}} 16\sin^2 u\cos u\,\left[-\cos v\right]\Big|_u^{\pi-u}\,du\\
&= \int_0^{\frac{\pi}{2}} 16\sin^2 u\cos u\,[2\cos u]\,du = 8\int_0^{\frac{\pi}{2}}\sin^2 2u\,du = 4\int_0^{\frac{\pi}{2}} 1-\cos 4u\,du = 2\pi.
\end{aligned}$$

評分標準:

1. 若直接計算線積分，寫出參數 $\mathbf{r}(t)$ 、 計算 $\mathbf{F}, \mathbf{r}'(t)$ 並代入 $\int_C \mathbf{F} \cdot d\mathbf{r}$ 正確得7分，後續計算正確再得6分。 但若 $\mathbf{r}(t), \mathbf{F}, \mathbf{r}'(t)$ 中有錯誤、計算流程正確本題最多得7分。
2. 若使用 Stokes 定理則，寫出參數 $\mathbf{r}(u,v)$、 計算 $\text{curl}\mathbf{F}, \mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_u \times \mathbf{r}_v$ (或 $\mathbf{n}, dS$) 並代入 $\int_S \text{curl}\mathbf{F} \cdot d\mathbf{S}$ 正確得8分，後續計算正確再得5分。 但若 $\text{curl}\mathbf{F}, \mathbf{r}_u, \mathbf{r}_v, \mathbf{r}_u \times \mathbf{r}_v$ 中有任何錯誤、計算流程正確本題最多得8分。

9. (15%) Consider the vector field $\mathbf{F}(\mathbf{x}) = \dfrac{\mathbf{x}}{|\mathbf{x}|^3}$, that is,

$$\mathbf{F}(x,y,z) = \frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\,\mathbf{k}.$$

(a) Evaluate $\iint_{S_1} \mathbf{F} \bullet d\mathbf{S}$, where S_1 is the part of the sphere $x^2+y^2+z^2=1$ inside the cone $z = \sqrt{\dfrac{x^2+y^2}{3}}$ with upward orientation.

(b) Evaluate $\iint_{S_2} \mathbf{F} \bullet d\mathbf{S}$, where S_2 is the part of the cone $z = \sqrt{\dfrac{x^2+y^2}{3}}$ between planes $z = \dfrac{1}{2}$ and $z = \dfrac{2}{\sqrt{3}}$ with outward orientation.

(c) Use the Divergence Theorem to evaluate $\iint_{S_3} \mathbf{F} \bullet d\mathbf{S}$, where S_3 is the part of the paraboloid $z = \dfrac{6-x^2-y^2}{\sqrt{3}}$ inside the cone $z = \sqrt{\dfrac{x^2+y^2}{3}}$ with upward orientation.

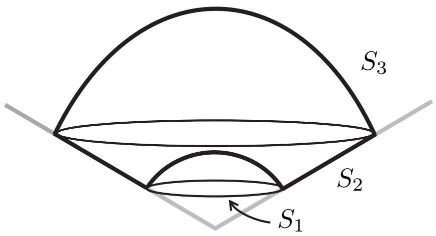

Figure 4: Parts of sphere, cone, and paraboloid.

Solution:

(a) Method 1:

Let $D_1 = \{(u\ ,v\ ) \mid u^2+v^2 \le (\frac{\sqrt{3}}{2})^2\}$. Then S_1 can be deseribed as

$$r(u,v) = (u\ ,v\ ,\sqrt{1-u^2-v^2}\ ) \text{ on } D_1. \textbf{ (2 pts)}$$

So $\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{D_1} \mathbf{F}(r(u,v)) \cdot (r_u \times r_v)\ dA$. Note that

$$\mathbf{F}(r(u,v)) = r(u,v)$$
$$r_u \times r_v = (\frac{u}{\sqrt{1-u^2-v^2}}\ ,\frac{v}{\sqrt{1-u^2-v^2}}\ ,1\ ) \leftarrow \textbf{(2 pts)}$$

Thus

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{D_1} \frac{1}{\sqrt{1-u^2-v^2}}\ dA = \int_0^{2\pi}\int_0^{\sqrt{3}/2} \frac{r}{\sqrt{1-r^2}}\ dr d\theta = \pi. \textbf{ (2 pts)}$$

Method 2:

Observe that S_1 can be desribed as

$$r(u,v) = (\sin u \cos v\ ,\sin u \sin v\ ,\cos u) \textbf{ (2 pts)}$$

for $0 \le u \le \pi/3,\ 0 \le v \le 2\pi$. Since

$$\mathbf{F}(r(u,v)) = r(u,v)$$
$$r_u \times r_v = \sin u \cdot r(u,v), \leftarrow \textbf{(2 pts)}$$

we obtain

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^{\pi/3} r(u,v) \cdot \sin u \cdot r(u,v)\ dudv = \pi\ \textbf{(2 pts)}$$

Method 3:

Observe that $\text{div}(\mathbf{F}) = 0$. So by divergence theorem,

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iint_{R_1} F \cdot d\mathbf{S},$$

where R_1 is the surface

$$r(u,v) = (u\cos v\ , u\sin v\ , \frac{1}{2}\ )\ \textbf{(2 pts)}$$

for $0 \le u \le \frac{\sqrt{3}}{2}$, $0 \le v \le 2\pi$. Note that

$$\mathbf{F}(r(u,v)) = (u^2 + 1/4)^{-3/2} \cdot r(u,v)$$
$$r_u \times r_v = (0\ , 0\ , u\ ) \leftarrow \textbf{(2 pts)}$$

Thus

$$\iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \frac{1}{2} \int_0^{2\pi} \int_0^{\sqrt{3}/2} \frac{u}{(u^2 + 1/4)^{3/2}}\ dudv = \pi.\ \textbf{(2 pts)}$$

(b) Method 1:

Let $D_2 = \{(u\ , v\ ) \mid (\frac{\sqrt{3}}{2})^2 \le u^2 + v^2 \le (2)^2\}$. Then S_2 can be described as

$$r(u,v) = (u\ , v\ , \sqrt{\frac{u^2 + v^2}{3}}\ ) \text{ on } D_2.\ \textbf{(2 pts)}$$

Note that

$$\mathbf{F}(r(u,v)) = (\frac{4}{3}(u^2 + v^2))^{-3/2} \cdot r(u,v)$$
$$r_u \times r_v = (3(u^2 + v^2))^{-1/2} \cdot (-u\ , -v\ , \sqrt{3(u^2 + v^2)}\ ) \leftarrow \textbf{(2 pts)}$$

So

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{D_2} 0\ dA = 0.\ \textbf{(1 pt)}$$

Method 2:

Observe that S_2 can be described as

$$r(u,v) = (u\cos v\ , u\sin v\ , \frac{u}{\sqrt{3}}\ )\ \textbf{(2 pts)}$$

for $\frac{\sqrt{3}}{2} \le u \le 2$, $0 \le v \le 2\pi$. Note that

$$\mathbf{F}(r(u,v)) = (\frac{4}{3}u^2)^{-3/2} \cdot r(u,v)$$
$$r_v \times r_u = (\frac{1}{\sqrt{3}}u\cos v\ , \frac{1}{\sqrt{3}}u\sin v\ , -u) \leftarrow \textbf{(2 pts)}$$

So

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint F(r(u,v)) \cdot (r_v \times r_u)\ dA = 0\ \textbf{(1 pt)}$$

Method 3:

Observe that $\text{div}(\mathbf{F}) = 0$. So by divergence theorem,

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \iint_{R_1} F \cdot d\mathbf{S} - \iint_{R_2} F \cdot d\mathbf{S},$$

where R_1 is as in method 3, (a), and R_2 is the surface

$$r(u,v) = (u\cos v \ , u\sin v \ , \frac{2}{\sqrt{3}}) \ \textbf{(2 pts)}$$

for $0 \le u \le 2$, $0 \le v \le 2\pi$. Note that

$$\mathbf{F}(r(u,v)) = (u^2 + 4/3)^{-3/2} \cdot r(u,v)$$
$$r_u \times r_v = (0 \ , 0 \ , u \ ) \leftarrow \textbf{(2 pts)}$$

So

$$\iint_{R_2} F \cdot d\mathbf{S} = \frac{2}{\sqrt{3}} \int_0^{2\pi} \int_0^2 \frac{u}{(u^2+4/3)^{3/2}} \, dudv = \pi, \ \textbf{(1 pt)}$$

i.e.

$$\iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \pi - \pi = 0$$

(you will lose 1 pt if your answer is not 0 in this step)

(c) **Method 1:**

Observe that, by the divergence theorem,

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} + \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} = \iiint_{S_1+S_2+S_3} \text{div}(F) \, dV.$$

↑**(1 pt)** ↑**(1 pt)**

Since $\text{div}(\mathbf{F}) = 0$ **(2 pts)** by a simple calculation, we obtain

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \iint_{S_1} \mathbf{F} \cdot d\mathbf{S} - \iint_{S_2} \mathbf{F} \cdot d\mathbf{S} = \pi - 0 = \pi.$$

Method 2:

Since $\text{div}(\mathbf{F}) = 0$ **(2 pts)**, we have

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \iint_{R_2} F \cdot d\mathbf{S},$$

where R_2 is as in method 3, (b). Now again by method 3, (b), we obtain

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = \iint_{R_2} F \cdot d\mathbf{S} = \pi. \ \textbf{(2 pts)}$$

Method 3:

Observe that S_3 can be described as

$$r(u,v) = (u\cos v \ , u\sin v \ , \frac{6-u^2}{\sqrt{3}}). \ \textbf{(2 pts)}$$

for $0 \le u \le 2$, $0 \le v \le 2\pi$. Note that

$$\mathbf{F}(r(u,v)) = (\frac{u^4 - 9u^2 + 36}{3})^{-3/2} \cdot r(u,v)$$
$$r_u \times r_v = (\frac{2u^2\cos v}{\sqrt{3}} \ , \frac{2u^2\cos v}{\sqrt{3}} \ , u).$$

Thus

$$\iint_{S_3} \mathbf{F} \cdot d\mathbf{S} = 6\pi \int_0^2 \frac{u^3 + 6u}{(u^4 - 9u^2 + 36)^{3/2}} \, du = \pi \ \textbf{(2 pts)}$$