微乙小考五 (2016/12/29)

1. (7分) 計算曲線 $y = \dfrac{e^x + e^{-x}}{2}$ 在 $-1 \le x \le 1$ 範圍的曲線長度。

sol: $y(x) = \cosh(x)$, $y'(x) = \sinh(x)$.

$$\begin{aligned}\int_{-1}^{1} \sqrt{1 + [y'(x)]^2}\ dx &= \int_{-1}^{1} \sqrt{1 + \sinh^2(x)}\ dx \\ &= \int_{-1}^{1} \cosh(x)\ dx = \sinh(x)\Big|_{-1}^{1} = 2\sinh(1) = e + e^{-1}.\end{aligned}$$

2. (6分) 求 $x^2 \ln(1+x)$ 在 $x = 0$ 的 n 次泰勒多項式。

sol: For $x \in (-1, 1)$,

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-x)^k.$$

The series converge uniformly on any closed interval inside $(-1, 1)$. Hence by using term-by-term integration, for $x \in (-1, 1)$,

$$\int_0^x \frac{dt}{1+t} = \int_0^x \sum_{k=0}^{\infty} (-t)^k\ dt = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+1}}{k+1} - 0.$$

We have

$$x^2 \ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{k+3}}{k+1}.$$

So the n-th polynomial of $x^2 \ln(1+x)$ is

$$P_n(x) = \begin{cases} 0 & n < 3 \\ \displaystyle\sum_{k=0}^{n-3} \frac{(-1)^k x^{k+3}}{k+1} & n \ge 3. \end{cases}$$

3. (7分) 令 Ω 為曲線 $x = 0$, $x = 1$, $y = 0$ 以及 $y = (x+1)\sqrt{\dfrac{x^3}{3} + \dfrac{x^2}{2}}$ 所圍成的區域。求Ω 繞 y 軸旋轉的旋轉體體積。

sol: (Shell method) The height at x is $h = (x+1)\sqrt{\dfrac{x^3}{3} + \dfrac{x^2}{2}} - 0$. Integrate the volume of each slight shell:

$$\int_0^1 2\pi x \left[(x+1)\sqrt{\frac{x^3}{3} + \frac{x^2}{2}}\right]\ dx. \tag{*}$$

By observing $u = \dfrac{x^3}{3} + \dfrac{x^2}{2}$, $du = (x^2 + x)dx$, we have

$$(*) = \int_{x=0}^{x=1} 2\pi\sqrt{u}\ du = 2\pi \frac{2}{3} u^{3/2}\Big|_{x=0}^{x=1} = \frac{4\pi}{3}\left(\frac{5}{6}\right)^{3/2}.$$