1051微甲01-04班期末考解答和評分標準

1. (12%) Evaluate the integrals.

(a) $\displaystyle\int \frac{1}{\sin x\cos^2 x}\mathrm{d}x.$ (b) $\displaystyle\int \tan^{-1}(\sqrt{x})\mathrm{d}x.$

Solution:

(a) **Method 1.**

$$\begin{aligned}\int \frac{1}{\sin x\cos^2 x}\mathrm{d}x &= \int \frac{\sin x}{\sin^2 x\cos^2 x}\mathrm{d}x\\ &(\text{Let } u=\cos x,\ \mathrm{d}u=-\sin x\mathrm{d}x.)\\ &= \int \frac{-\mathrm{d}u}{(1-u^2)u^2} \qquad (1\%)\\ &= \int \frac{\mathrm{d}u}{u^2(u-1)(u+1)}\\ &= \int \frac{-1}{u^2} + \Gamma\frac{1/2}{u-1} + \frac{-1/2}{u+1}\mathrm{d}u \qquad (3\%)\\ &= \frac{1}{u} + \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + C\\ &= \sec x + \frac{1}{2}\ln\left|\frac{\cos x-1}{\cos x+1}\right| + C \qquad (2\%)\end{aligned}$$

Method 2.

$$\begin{aligned}\int \frac{1}{\sin x\cos^2 x}\mathrm{d}x &= \int \csc x\sec^2 x\mathrm{d}x\\ &\begin{pmatrix}\text{Let } u=\csc x, & \mathrm{d}v=\sec^2\mathrm{d}x\\ \mathrm{d}u=-\csc x\cot x\mathrm{d}x, & v=\tan x.\end{pmatrix} \qquad (1\%)\\ &= \csc x\tan x - \int \tan x(-\csc x\cot x)\mathrm{d}x \qquad (2\%)\\ &= \sec x + \int \csc x\mathrm{d}x\\ &= \sec x - \ln|\csc x+\cot x| + C \qquad (3\%)\end{aligned}$$

Method 3.

$$\text{Let } t=\tan(\frac{x}{2}). \Rightarrow \sin x = \frac{2t}{1+t^2},\ \cos x=\frac{1-t^2}{1+t^2},\ \mathrm{d}x=\frac{2}{1+t^2}\mathrm{d}t. \qquad (1\%)$$

$$\begin{aligned}\int \frac{1}{\sin x\cos^2 x}\mathrm{d}x &= \int \frac{1}{\left(\frac{2t}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)^2}\frac{2}{1+t^2}\mathrm{d}t\\ &= \int \frac{(1+t^2)^2}{t(1-t^2)^2}\mathrm{d}t\\ &= \int \frac{1}{t} + \frac{1}{(t-1)^2} - \frac{1}{(t+1)^2}\mathrm{d}t \qquad (3\%)\\ &= \ln|\tan(\frac{x}{2})| - \frac{1}{\tan(\frac{x}{2})-1} + \frac{1}{\tan(\frac{x}{2})+1} + C \qquad (2\%)\end{aligned}$$

Method 4.

$$\begin{aligned}\int \frac{1}{\sin x\cos^2 x}\mathrm{d}x &= \int \frac{\sin^2 x+\cos^2 x}{\sin x\cos^2 x}\mathrm{d}x \qquad (1\%)\\ &= \int \frac{\sin x}{\cos^2 x}\mathrm{d}x + \int \frac{1}{\sin x}\mathrm{d}x\\ &= \int \frac{-\mathrm{d}(\cos x)}{\cos^2 x} + \int \csc x\mathrm{d}x \qquad (2\%)\\ &= \frac{1}{\cos x} - \ln|\csc x+\cot x| + C \qquad (3\%)\end{aligned}$$

評分標準:

- 有正確寫出 Partial Fraction 但是係數算錯扣 1 分
- 最後答案沒有標示把變數換回x 或 ln沒加絕對值或沒有 $+C$ 扣 1 分

(b) **Method 1.**

$$\text{Let } u = \sqrt{x},\ \mathrm{d}u = \frac{1}{2\sqrt{x}}\mathrm{d}x. \qquad (2\%)$$

$$\begin{aligned}\int \tan^{-1}(\sqrt{x})\mathrm{d}x &= \int 2u\tan^{-1} u\mathrm{d}u \\ &= u^2\tan^{-1} u - \int \frac{u^2}{1+u^2}\mathrm{d}u \qquad (3\%) \\ &= u^2\tan^{-1} u - \int \left(1 - \frac{1}{1+u^2}\right)\mathrm{d}u \\ &= u^2\tan^{-1} u - u + \tan^{-1} u + C \\ &= (x+1)\tan^{-1}(\sqrt{x}) - \sqrt{x} + C \qquad (1\%)\end{aligned}$$

Method 2.

$$\text{Let } \sqrt{x} = \tan\theta.\ \Longrightarrow\ x = \tan^2\theta,\ \mathrm{d}x = 2\tan\theta\sec^2\theta\mathrm{d}\theta. \qquad (2\%)$$

$$\begin{aligned}\int \tan^{-1}(\sqrt{x})\mathrm{d}x &= \int \theta\mathrm{d}(\tan^2\theta) \\ &= \theta\tan^2\theta - \int \tan^2\theta\mathrm{d}\theta \\ &= \theta\tan^2\theta - \int (\sec^2\theta - 1)\mathrm{d}\theta \\ &= \theta\tan^2\theta - \tan\theta + \theta + C \qquad (3\%) \\ &= (x+1)\tan^{-1}(\sqrt{x}) - \sqrt{x} + C \qquad (1\%)\end{aligned}$$

評分標準:

- 最後答案沒有標示把變數換回x 或答案沒有 $+C$ 扣 1 分
- 微分計算算錯扣 2 分

2. (12%) Evaluate the integrals.

(a) $\int x\sqrt{8+2x-x^2}\mathrm{d}x$ (b) $\int \frac{x^2-1}{(x^2+2x+2)^2}\mathrm{d}x$

Solution:

(a) Let $x = 1 + 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta d\theta$. Thus we have

$$
\begin{aligned}
& \int x\sqrt{8+2x-x^2}dx \\
= & \int x\sqrt{9-(x-1)^2}dx \\
= & \int (1+3\sin\theta)\cdot 3\cos\theta \cdot 3\cos\theta d\theta \quad \textbf{(2pts)} \\
= & \int (9\cos^2\theta + 27\sin\theta\cos^2\theta)d\theta \\
= & \int \left(9\cdot\frac{1+\cos(2\theta)}{2} + 27\sin\theta\cos^2\theta\right)d\theta \\
= & 9\Big(\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\Big) + 27\cdot\frac{-1}{3}\cos^3\theta + C \quad \textbf{(2pts)} \\
= & \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta - 9\cos^3\theta + C \\
= & \frac{9}{2}\sin^{-1}\left(\frac{x-1}{3}\right) + \frac{1}{2}(x-1)\sqrt{8+2x-x^2} - \frac{1}{3}(8+2x-x^2)^{\frac{3}{2}} + C \quad \textbf{(2pts)}
\end{aligned}
$$

(b) Let $x = -1 + \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta d\theta$. Thus we have

$$
\begin{aligned}
& \int \frac{x^2-1}{(x^2+2x+2)^2}dx \\
= & \int \frac{x^2-1}{((x+1)^2+1)^2}dx \\
= & \int \frac{(\tan\theta-1)^2-1}{\sec^4\theta}\cdot\sec^2\theta d\theta \quad \textbf{(2pts)} \\
= & \int (\sin^2\theta - 2\sin\theta\cos\theta)d\theta \\
= & \frac{1}{2}\theta - \frac{1}{4}\sin(2\theta) + \cos^2\theta + C \quad \textbf{(2pts)} \\
= & \frac{1}{2}\tan^{-1}(x+1) - \frac{1}{2}\cdot\frac{x+1}{x^2+2x+2} + \frac{1}{x^2+2x+2} + C \\
= & \frac{1}{2}\tan^{-1}(x+1) - \frac{x-1}{2(x^2+2x+2)} + C \quad \textbf{(2pts)}
\end{aligned}
$$

3. (10%) Find the reduction formula $I_n = \int (\ln x)^n \mathrm{d}x$, where n is a positive number.(i.e. write I_n in terms of I_{n-1}.) Evaluate the improper integral $\int_0^1 (\ln x)^n \mathrm{d}x$ or explain why it is divergent.

Solution:

$$\text{Let } u = (\ln x)^n, \qquad \mathrm{d}v = \mathrm{d}x$$
$$\mathrm{d}u = n(\ln x)^{n-1}\frac{1}{x}\mathrm{d}x, \qquad v = x.$$
$$\int (\ln x)^n \mathrm{d}x = x(\ln x)^n - n\int (\ln x)^{n-1}\mathrm{d}x \qquad (2\%)$$

Therefore,

$$I_n = x(\ln x)^n - nI_{n-1}.$$

$$\begin{aligned}\int_t^1 (\ln x)^n \mathrm{d}x &= x(\ln x)^n\Big|_t^1 - n\int_t^1 (\ln x)^{n-1}\mathrm{d}x \\ &= 1\cdot(\ln 1)^n - t(\ln t)^n - n\int_t^1 (\ln x)^{n-1}\mathrm{d}x \\ &= -t(\ln t)^n - n\int_t^1 (\ln x)^{n-1}\mathrm{d}x.\end{aligned}$$

Claim: The improper integral $\int_0^1 (\ln x)^n \mathrm{d}x$ converges for all $n \in \mathbb{N}$. **Proof.** We prove the claim by mathematical induction. First, consider the case $n = 1$,

$$\begin{aligned}\int_0^1 (\ln x)\mathrm{d}x &= \lim_{t\to 0^+}\int_t^1 (\ln x)\mathrm{d}x \\ &= \lim_{t\to 0^+} -t(\ln t) - 1 \\ &= \lim_{t\to 0^+} -\frac{\ln t}{(\frac{1}{t})} - 1 \overset{\text{L'Hospital's rule}}{=} \lim_{t\to 0^+} -\frac{\frac{1}{t}}{\frac{-1}{t^2}} - 1 \\ &= \lim_{t\to 0^+} t - 1 = -1 \text{ converges.} \qquad (2\%)\end{aligned}$$

Suppose that for $n = k-1$, the improper integral $\int_0^1 (\ln x)^{k-1}\mathrm{d}x$ converges. Consider the case $n = k$, we have

$$\begin{aligned}\int_0^1 (\ln x)^k \mathrm{d}x &= \lim_{t\to 0^+}\int_t^1 (\ln x)^k \mathrm{d}x \\ &= \lim_{t\to 0^+}\left(-t(\ln t)^k - k\int_t^1 (\ln x)^{k-1}\mathrm{d}x\right).\end{aligned}$$

By L'Hospital's rule, the limit

$$\lim_{t\to 0^+}\frac{\ln t}{t^{-1/k}} = \lim_{t\to 0^+}\frac{\frac{1}{t}}{\frac{-1}{k}t^{-\frac{1}{k}-1}} = \lim_{t\to 0^+} -kt^{\frac{1}{k}} = 0$$

since k is a positive integer implies that $\frac{1}{k} > 0$.
Hence, the limit

$$\lim_{t\to 0^+} -t(\ln t)^k = \lim_{t\to 0^+} -\frac{(\ln t)^k}{\left(\frac{1}{t^{1/k}}\right)^k} = \lim_{t\to 0^+} -\left(\frac{\ln t}{t^{-1/k}}\right)^k = -\left(\lim_{t\to 0^+}\frac{\ln t}{t^{-1/k}}\right)^k = 0, \qquad (3\%)$$

and the improper integral

$$\begin{aligned}\int_0^1 (\ln x)^k \mathrm{d}x &= \lim_{t\to 0^+}\left(-t(\ln t)^k - k\int_t^1 (\ln x)^{k-1}\mathrm{d}x\right) \\ &= \left(\lim_{t\to 0^+} -t(\ln t)^k - k\lim_{t\to 0^+}\int_t^1 (\ln x)^{k-1}\mathrm{d}x\right) \\ &= -k\int_0^1 (\ln x)^{k-1}\mathrm{d}x \text{ converges.}\end{aligned}$$

Therefore, by mathematical induction, we have shown that the improper integral $\int_0^1 (\ln x)^n \mathrm{d}x$ converges for all $n \in \mathbb{N}$, and it converges to

$$\begin{aligned}\int_0^1 (\ln x)^n \mathrm{d}x &= -n \int_0^1 (\ln x)^{n-1} \mathrm{d}x = -n[-(n-1)] \int_0^1 (\ln x)^{n-2} \mathrm{d}x \\ &= ... = -n[-(n-1)][-(n-2)][-(n-3)]...(-3)(-2) \int_0^1 (\ln x) \mathrm{d}x \\ &= (-1)^n n! \qquad (3\%)\end{aligned}$$

評分標準:

- $\int_0^1 (\ln x)^n \mathrm{d}x$ **沒有照定義取右極限** $\lim\limits_{t \to 0^+} \int_t^1 (\ln x)^n \mathrm{d}x$ **扣** 1 **分**
- **沒有說明怎麼計算** $\lim\limits_{t \to 0^+} -t(\ln t)^n = 0$ **的過程扣** 2 **分**

4. (12%) Find the values of a and b for which the improper integral

$$\int_0^\infty \frac{e^{-ax}}{x^b(1+x^2)}\mathrm{d}x = \int_0^1 \frac{e^{-ax}}{x^b(1+x^2)}\mathrm{d}x + \int_1^\infty \frac{e^{-ax}}{x^b(1+x^2)}\mathrm{d}x \text{ converges.}$$

Hint: Discuss cases $a = 0$ and $a \neq 0$, respectively.

Solution:

(1) (4pt) If $a = 0$, we have

$$\int_0^\infty \frac{1}{x^b(1+x^2)}dx = \int_0^1 \frac{1}{x^b(1+x^2)}dx + \int_1^\infty \frac{1}{x^b(1+x^2)}dx$$

On the interval $[0,1]$, since $\frac{1}{2} \leq \frac{1}{1+x^2} \leq 1$, we have

$$\frac{1}{2x^b} \leq \frac{1}{x^b(1+x^2)} \leq \frac{1}{x^b}.$$

Since $\int_0^1 \frac{1}{x^b}dx$ is convergent if and only if $b < 1$ (notice that the integral $\int_0^1 \frac{1}{x^b(1+x^2)}dx$ is a definite integral if $b \leq 0$, so it is a finite value), we get $\int_0^1 \frac{1}{x^b(1+x^2)}dx$ is convergent if and only if $b < 1$ by the convergent theorem.

On the interval $[1,\infty)$, since $x^{b+2} \geq x^b$, we have

$$\frac{1}{2x^{b+2}} \leq \frac{1}{x^b(1+x^2)} \leq \frac{1}{x^{b+2}}.$$

Since the improper integral $\int_1^\infty \frac{1}{x^{b+2}}dx$ is convergent if and only if $b+2 > 1$ (that is, $b > -1$), $\int_1^\infty \frac{1}{x^b(1+x^2)}dx$ is convergent if and only if $b > -1$.

Therefore, if $a = 0$ and $-1 < b < 1$, the improper integral is convergent.

(2) (4pt) If $a < 0$, since $\lim_{x\to\infty} \frac{e^{-ax}}{x^b(1+x^2)} = \infty$, the improper intergral is divergent for any $b \in \mathbb{R}$.

(3) (4pt) If $a > 0$, since $\frac{e^{-ax}}{1+x^2}$ is continuous on $[0,1]$ by the Extreme Value Theorem, there exist m and M such that $m \leq \frac{e^{-ax}}{1+x^2} \leq M$, so

$$\frac{m}{x^b} \leq \frac{e^{-ax}}{x^b(1+x^2)} \leq \frac{M}{x^b}.$$

Since $\int_0^1 \frac{1}{x^b}dx$ is convergent if and only if $b < 1$, we get $\int_0^1 \frac{1}{x^b(1+x^2)}dx$ is convergent if and only if $b < 1$ by the comparison theorem.

On the interval $[1,\infty)$, since $\lim_{x\to\infty} \frac{e^{-ax}}{x^b(1+x^2)} = 0$ for any $b \in \mathbb{R}$, there exists x_0 such that for all $x > x_0$, we have $\frac{e^{-\frac{a}{2}}}{x^b(1+x^2)} < 1$.

$$0 \leq \frac{e^{-ax}}{x^b(1+x^2)} = e^{-\frac{a}{2}x} \cdot \frac{e^{-\frac{a}{2}x}}{x^b(1+x^2)} \leq e^{-\frac{a}{2}x},$$

and $\int_1^\infty e^{-\frac{a}{2}x}dx$ is convergent, $\int_1^\infty \frac{e^{-ax}}{x^b(1+x^2)}dx$ is convergent for all $b \in \mathbb{R}$.

Therefore, if $a > 0$ and $b < 1$, the improper integral is convergent.

5. (12%) Find the volume of Gulliver's Tunnel (格列佛隧道), which is half of the solids of revolution obtained by rotating the region bounded by $y = \frac{1}{1+e^{3x}}$, $y = 0$, $x = -\frac{2}{3}\ln 3$, and $x = \ln 2$, about the x-axis.

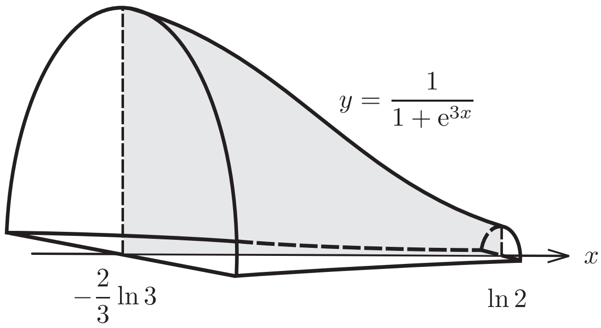


Figure 1: Find the volume of Gulliver's Tunnel.

Solution:

1. The volume by disk method is

$$\text{Volume } = \frac{\pi}{2}\int_{-\frac{2}{3}\ln 3}^{\ln 2}\left(\frac{1}{1+e^{3x}}\right)^2 dx \text{ (5pts)} .$$

Let $u = e^{3x}$. Then $du = 3e^{3x}\,dx$ or $dx = \frac{1}{3u}\,du$ (1pt). The upper limit is $u = 8$ and the lower limit is $u = \frac{1}{9}$ (1pt). Thus

$$\text{Volume } = \frac{\pi}{6}\int_{\frac{1}{9}}^{8}\frac{1}{u(1+u)^2}\,du = \frac{\pi}{6}\int_{\frac{1}{9}}^{8}\frac{A}{u} + \frac{B}{1+u} + \frac{C}{(1+u)^2}\,du \text{ (2pts)} .$$

From $A(1+u)^2 + Bu(1+u) + Cu = 1$ we determine $A = 1$, $B = -1$ and $C = -1$ and so

$$\begin{aligned}\text{Volume } &= \frac{\pi}{6}\int_{\frac{1}{9}}^{8}\frac{1}{u} - \frac{1}{1+u} - \frac{1}{(1+u)^2}\,du = \frac{\pi}{6}\left[\ln u - \ln(1+u) + \frac{1}{1+u}\right]_{\frac{1}{9}}^{8} \\ &= \frac{\pi}{6}\left(\ln 8 + \ln 9 - \ln 9 + \ln\frac{10}{9} + \frac{1}{9} - \frac{9}{10}\right) = \frac{\pi}{6}\left(\ln\frac{80}{9} - \frac{71}{90}\right) \text{ (3pts)} .\end{aligned}$$

2. The volume by disk method is

$$\text{Volume } = \frac{\pi}{2}\int_{-\frac{2}{3}\ln 3}^{\ln 2}\left(\frac{1}{1+e^{3x}}\right)^2 dx \text{ (5pts)} .$$

Let $u = \frac{1}{1+e^{3x}}$, or $x = \frac{1}{3}\ln\left(\frac{1}{u} - 1\right)$. Then $dx = \frac{1}{3(u^2-u)}\,du$ (1pt). The upper limit is $u = \frac{1}{9}$ and the lower limit is $u = \frac{9}{10}$ (1pt). Thus

$$\begin{aligned}\text{Volume } &= \frac{\pi}{6}\int_{\frac{9}{10}}^{\frac{1}{9}}\frac{u}{u-1}\,du = \frac{\pi}{6}\int_{\frac{9}{10}}^{\frac{1}{9}} 1 + \frac{1}{u-1}\,du \text{ (2pts)} \\ &= \frac{\pi}{6}[u + \ln|u-1|]_{\frac{9}{10}}^{\frac{1}{9}} = \frac{\pi}{6}\left(\frac{1}{9} - \frac{9}{10} + \ln\frac{8}{9} - \ln\frac{1}{10}\right) = \frac{\pi}{6}\left(\ln\frac{80}{9} - \frac{71}{90}\right) \text{ (3pts)} .\end{aligned}$$

3. The volume by disk method is

$$\text{Volume } = \frac{\pi}{2}\int_{-\frac{2}{3}\ln 3}^{\ln 2}\left(\frac{1}{1+e^{3x}}\right)^2 dx \text{ (5pts)} .$$

Let $\tan\theta = e^{\frac{3}{2}x}$. Then $\sec^2\theta\, d\theta = \frac{3}{2}e^{\frac{3}{2}x}\, dx$, or $dx = \frac{2}{3}\frac{\sec^2\theta}{\tan\theta}\, d\theta$ (1pt). The upper limit is $\theta = \tan^{-1}\sqrt{8}$ and the lower limit is $\theta = \tan^{-1}\frac{1}{3}$ (1pt). Thus

$$\begin{aligned}
\text{Volume } &= \frac{\pi}{2}\int_{\tan^{-1}\frac{1}{3}}^{\tan^{-1}\sqrt{8}} \frac{1}{\sec^4\theta}\frac{2}{3}\frac{\sec^2\theta}{\tan\theta}\, d\theta = \frac{\pi}{3}\int_{\tan^{-1}\frac{1}{3}}^{\tan^{-1}\sqrt{8}} \frac{\cos^3\theta}{\sin\theta}\, d\theta \text{ (1pts)}\\
&= \frac{\pi}{3}\int_{\tan^{-1}\frac{1}{3}}^{\tan^{-1}\sqrt{8}} \frac{1-\sin^2\theta}{\sin\theta}\, d(\sin\theta) \text{ (2pts) } = \frac{\pi}{3}\left[\ln(\sin\theta) - \frac{1}{2}\sin^2\theta\right]_{\tan^{-1}\frac{1}{3}}^{\tan^{-1}\sqrt{8}} \text{ (2pts)}\\
&= \frac{\pi}{3}\left(\ln\frac{\sqrt{8}}{3} - \ln\frac{1}{\sqrt{10}} - \frac{1}{2}\frac{8}{9} + \frac{1}{2}\frac{1}{10}\right) = \frac{\pi}{6}\left(\ln\frac{80}{9} - \frac{71}{90}\right).
\end{aligned}$$

4. From $y = \frac{1}{1+e^{3x}}$ we have $x = \frac{1}{3}\ln\left(\frac{1}{y} - 1\right)$ (1pt). At $x = \ln 2$, $y = \frac{1}{9}$ and at $x = -\frac{2}{3}\ln 3$, $y = \frac{9}{10}$ (1pt). Thus the volume by cylindrical shell method is

$$\text{Volume } = \frac{1}{2}\left[2\pi\int_0^{\frac{1}{9}} y\left(\ln 2 - (-\frac{2}{3}\ln 3)\right)\, dy + 2\pi\int_{\frac{1}{9}}^{\frac{9}{10}} y\left(\frac{1}{3}\ln\left(\frac{1}{y}-1\right) - (-\frac{2}{3}\ln 3)\right)\, dy\right] \text{ (5pts) .}$$

We integrate by parts to get

$$\begin{aligned}
\int y\ln\left(\frac{1}{y}-1\right)\, dy &= \frac{y^2}{2}\ln\left(\frac{1}{y}-1\right) - \int \frac{y^2}{2}\frac{1}{y^2-y}\, dy \text{ (3pts)}\\
&= \frac{y^2}{2}\ln\left(\frac{1}{y}-1\right) - \frac{1}{2}\int 1 + \frac{1}{y-1}\, dy \text{ (1pts)}\\
&= \frac{y^2}{2}\ln\left(\frac{1}{y}-1\right) - \frac{y}{2} - \frac{1}{2}\ln|y-1| \text{ (1pts)}
\end{aligned}$$

Thus

$$\begin{aligned}
\text{Volume } &= \frac{1}{2}\left[2\pi\int_0^{\frac{1}{9}} y\left(\ln 2 - (-\frac{2}{3}\ln 3)\right)\, dy + 2\pi\int_{\frac{1}{9}}^{\frac{9}{10}} y\left(\frac{1}{3}\ln\left(\frac{1}{y}-1\right) - (-\frac{2}{3}\ln 3)\right)\, dy\right]\\
&= \pi\left(\ln 2 + \frac{2}{3}\ln 3\right)\int_0^{\frac{1}{9}} y\, dy + \frac{\pi}{3}\int_{\frac{1}{9}}^{\frac{9}{10}} y\ln\left(\frac{1}{y}-1\right)\, dy + \frac{2\pi}{3}\ln 3\int_{\frac{1}{9}}^{\frac{9}{10}} y\, dy\\
&= \pi\left(\ln 2 + \frac{2}{3}\ln 3\right)\left[\frac{y^2}{2}\right]_0^{\frac{1}{9}} + \frac{\pi}{3}\left[\frac{y^2}{2}\ln\left(\frac{1}{y}-1\right) - \frac{y}{2} - \frac{1}{2}\ln|y-1|\right]_{\frac{1}{9}}^{\frac{9}{10}} + \frac{2\pi}{3}\ln 3\left[\frac{y^2}{2}\right]_{\frac{1}{9}}^{\frac{9}{10}}\\
&= \frac{\pi}{6}\left(\ln\frac{80}{9} - \frac{71}{90}\right).
\end{aligned}$$

6. (12%)

(a) Find the orthogonal trajectories of the family of curves $y = \sqrt[3]{x^3 + c}$, where c is an arbitrary constant.

(b) Solve the initial-value problem

$$y' + (\tan x)y = \sec^3 x,\ \ y(0) = 1.$$

Solution:

(a) Differentiating $y = \sqrt[3]{x^3 + c}$ yields

$$\frac{dy}{dx} = \frac{1}{3}\frac{3x^2}{(x^3+c)^{\frac{2}{3}}} = \frac{x^2}{(x^3+c)^{\frac{2}{3}}}\ \text{(1pt)}\ = \frac{x^2}{y^2}\ \text{(1pt)}$$

Alternatively, differentiating $y^3 = x^3 + c$ yields the same result:

$$3y^2\frac{dy}{dx} = 3x^2 \quad \text{so} \quad \frac{dy}{dx} = \frac{x^2}{y^2}\ \text{(2pts)}$$

We want to find a family of curves $C : (x, y(x))$ such that

$$\frac{dy}{dx} = -\frac{y^2}{x^2}\ \text{(2pts)}$$

This is a separable equation, we compute

$$\frac{1}{y^2}\frac{dy}{dx} = -\frac{1}{x^2}(1pt) \Rightarrow \int \frac{1}{y^2}\frac{dy}{dx}\,dx = \int -\frac{1}{x^2}\,dx \Rightarrow -\frac{1}{y} = \frac{1}{x} + C\ \text{(1pt)}\ .$$

Thus the family of orthogonal trajectories of $y = \sqrt[3]{x^3 + c}$ is

$$\frac{1}{x} + \frac{1}{y} = C.$$

(b) The integrating factor is

$$e^{\int \tan x\,dx}\ \text{(2pts)}\ = e^{\ln|\sec x|}\ \text{(1pt)}\ = \sec x\ \text{(you can pick any sort of integrating factor)}$$

Multiply the integrating factor we get

$$\sec x y' + \sec x \tan x \cdot y = \sec^4 x \Rightarrow \frac{d}{dx}(\sec x \cdot y) = \sec^4 x$$

Thus

$$\begin{aligned}\sec x \cdot y &= \int \sec^4 x\,dx\ \text{(1pt)}\\ &= \sec^2 x\,d\tan x = \int (\tan^2 x + 1)\,d\tan x = \frac{1}{3}\tan^3 x + \tan x + C\ \text{(1pt)}\end{aligned}$$

The initial condition $y(0) = 1$ implies $C = 1$ (1pt). Thus the solution is

$$y = \frac{1}{3}\cos x \tan^3 x + \sin x + \cos x.$$

Alternatively, integrating by parts yields

$$\begin{aligned}&\int \sec^4 x\,dx\\ &= \sec^2 x\,d\tan x = \sec^2 x \tan x - \int \tan x(2\sec^2 x \tan x)\,dx\\ &= \sec^2 x \tan x - 2\int \sec^2 x(\sec^2 x - 1)\,dx = \sec^2 x \tan x - 2\int \sec^4 x\,dx + 2\tan x\end{aligned}$$

Thus

$$\int \sec^4 x\,dx = \frac{1}{3}\sec^2 x \tan x + \frac{2}{3}\tan x + C\ \text{(1pt)}$$

The initial condition $y(0) = 1$ implies $C = 1$ (1pt). Thus the solution is

$$y = \frac{1}{3}\sec x \tan x + \frac{2}{3}\sin x + \cos x.$$

7. (12%) Consider the plum flower-like curve (梅花) as Figure 2. It is characterized by the polar equation

$$r = \frac{3}{2} + \cos\left(\frac{5}{2}\theta\right).$$

(a) Find the slopes of the tangent lines of the curve at the intersection point $P(r,\theta) = \left(\frac{3}{2}, \frac{\pi}{5}\right)$.

(b) Set up an integral that represents the length of the whole curve. You don't need to evaluate the integral.

(c) Find the area of the shaded region.

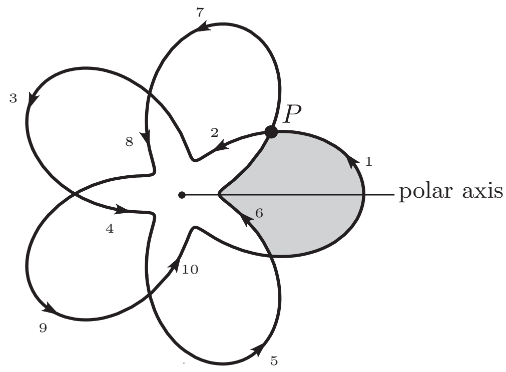
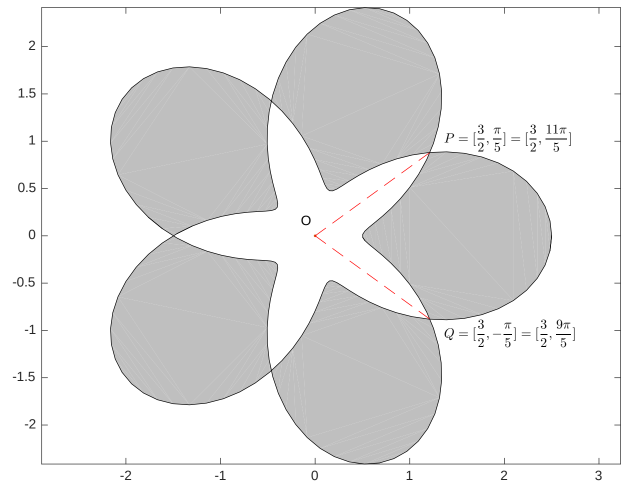


Figure 2: The plum flower-like curve.

Solution:



(a) **(method 1)**

Since $r = \frac{3}{2} + \cos\left(\frac{5}{2}\theta\right)$, we have $r' = -\frac{5}{2}\sin\left(\frac{5}{2}\theta\right)$.**(1%)**

Therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$**(1%)** $= \frac{-\frac{5}{2}\sin\left(\frac{5}{2}\theta\right)\sin\theta + \frac{3}{2}\cos\theta + \cos\left(\frac{5}{2}\theta\right)\cos\theta}{-\frac{5}{2}\sin\left(\frac{5}{2}\theta\right)\cos\theta - \frac{3}{2}\sin\theta - \cos\left(\frac{5}{2}\theta\right)\sin\theta}$

The slopes at $P = \left[\frac{3}{2}, \frac{\pi}{5}\right] = \left[\frac{3}{2}, \frac{11\pi}{5}\right]$ are:

$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{\pi}{5}}$ **(1%)** $= \frac{-\frac{5}{2}\sin\frac{\pi}{5} + \frac{3}{2}\cos\frac{\pi}{5}}{-\frac{5}{2}\cos\frac{\pi}{5} - \frac{3}{2}\sin\frac{\pi}{5}}$ and $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{11\pi}{5}}$ **(1%)** $= \frac{\frac{5}{2}\sin\frac{\pi}{5} + \frac{3}{2}\cos\frac{\pi}{5}}{\frac{5}{2}\cos\frac{\pi}{5} - \frac{3}{2}\sin\frac{\pi}{5}}$

(method 2)

$$x = r\cos\theta = \frac{3}{2}\cos\theta + \cos\left(\frac{5}{2}\theta\right)\cos\theta \left(\text{or } = \frac{3}{2}\cos\theta + \frac{1}{2}\cos\left(\frac{7}{2}\theta\right) + \frac{1}{2}\cos\left(\frac{3}{2}\theta\right)\right), \text{ and}$$

$$y = r\sin\theta = \frac{3}{2}\sin\theta + \cos\left(\frac{5}{2}\theta\right)\sin\theta \left(\text{or } = \frac{3}{2}\sin\theta + \frac{1}{2}\sin\left(\frac{7}{2}\theta\right) - \frac{1}{2}\sin\left(\frac{3}{2}\theta\right)\right)$$

Therefore, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}\theta}}{\frac{\mathrm{d}x}{\mathrm{d}\theta}} = \frac{-\frac{5}{2}\sin\left(\frac{5}{2}\theta\right)\sin\theta + \frac{3}{2}\cos\theta + \cos\left(\frac{5}{2}\theta\right)\cos\theta}{-\frac{5}{2}\sin\left(\frac{5}{2}\theta\right)\cos\theta - \frac{3}{2}\sin\theta - \cos\left(\frac{5}{2}\theta\right)\sin\theta}$ **(2%)** $\left(\text{or} = \frac{\frac{3}{2}\cos\theta + \frac{7}{4}\cos\left(\frac{7}{2}\theta\right) - \frac{3}{4}\cos\left(\frac{3}{2}\theta\right)}{-\frac{3}{2}\sin\theta - \frac{7}{4}\sin\left(\frac{7}{2}\theta\right) - \frac{3}{4}\sin\left(\frac{3}{2}\theta\right)}\right)$

The slopes at $P = \left[\frac{3}{2}, \frac{\pi}{5}\right] = \left[\frac{3}{2}, \frac{11\pi}{5}\right]$ are:

$\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{\pi}{5}}$ **(1%)** $= \frac{-\frac{5}{2}\sin\frac{\pi}{5} + \frac{3}{2}\cos\frac{\pi}{5}}{-\frac{5}{2}\cos\frac{\pi}{5} - \frac{3}{2}\sin\frac{\pi}{5}}$ and $\left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{11\pi}{5}}$ **(1%)** $= \frac{\frac{5}{2}\sin\frac{\pi}{5} + \frac{3}{2}\cos\frac{\pi}{5}}{\frac{5}{2}\cos\frac{\pi}{5} - \frac{3}{2}\sin\frac{\pi}{5}}$

$\left(\text{or } \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{\pi}{5}} = \frac{\frac{3}{2}\cos\frac{\pi}{5} + \frac{7}{4}\cos\frac{7\pi}{10} - \frac{3}{4}\cos\frac{3\pi}{10}}{-\frac{3}{2}\sin\frac{\pi}{5} - \frac{7}{4}\sin\frac{7\pi}{10} - \frac{3}{4}\sin\frac{3\pi}{10}} \text{ and } \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{\theta=\frac{11\pi}{5}} = \frac{\frac{3}{2}\cos\frac{1\pi}{5} - \frac{7}{4}\cos\frac{7\pi}{10} + \frac{3}{4}\cos\frac{3\pi}{10}}{-\frac{3}{2}\sin\frac{1\pi}{5} + \frac{7}{4}\sin\frac{7\pi}{10} + \frac{3}{4}\sin\frac{3\pi}{10}}\right)$

(*b*) $L = \int_0^{4\pi} \sqrt{r^2 + (r')^2}\mathrm{d}\theta$ **(2%)** $= \int_0^{4\pi} \sqrt{\left(\frac{3}{2} + \cos\left(\frac{5}{2}\theta\right)\right)^2 + \left(-\frac{5}{2}\sin\left(\frac{5}{2}\theta\right)\right)^2}\mathrm{d}\theta$

$\left(\text{or} = 10\int_0^{\frac{5}{2}\pi} \sqrt{\left(\frac{3}{2} + \cos\left(\frac{5}{2}\theta\right)\right)^2 + \left(-\frac{5}{2}\sin\left(\frac{5}{2}\theta\right)\right)^2}\mathrm{d}\theta = 2\int_0^{2\pi} \sqrt{\left(\frac{3}{2} + \cos\left(\frac{5}{2}\theta\right)\right)^2 + \left(-\frac{5}{2}\sin\left(\frac{5}{2}\theta\right)\right)^2}\mathrm{d}\theta\right)$

(*c*) By symmetry, the area will be

$$\text{Area} = 2\left(\int_0^{\frac{\pi}{5}} \frac{1}{2}\left[\frac{3}{2} + \cos\left(\frac{5}{2}\theta\right)\right]^2 \mathrm{d}\theta - \int_{2\pi}^{2\pi+\frac{5}{\pi}} \frac{1}{2}\left[\frac{3}{2} + \cos\left(\frac{5}{2}\theta\right)\right]^2 \mathrm{d}\theta\right) \textbf{(3\%)}$$

$$= \left(\int_0^{\frac{\pi}{5}} - \int_{2\pi}^{2\pi+\frac{\pi}{5}}\right)\left(\frac{9}{4} + 3\cos\left(\frac{5}{2}\theta\right) + \cos^2\left(\frac{5}{2}\theta\right)\right)\mathrm{d}\theta$$

$$= \left(\int_0^{\frac{\pi}{5}} - \int_{2\pi}^{2\pi+\frac{\pi}{5}}\right)\left(\frac{9}{4} + 3\cos\left(\frac{5}{2}\theta\right) + \frac{1+\cos(5\theta)}{2}\right)\mathrm{d}\theta$$

$$= \left[\frac{11}{4}\theta + \frac{6}{5}\sin\left(\frac{5}{2}\theta\right) \textbf{(1\%)} + \frac{1}{10}\sin(5\theta)\textbf{(1\%)}\right]\left(\Big|_0^{\frac{\pi}{5}} - \Big|_{2\pi}^{2\pi+\frac{\pi}{5}}\right) = \frac{12}{5}\textbf{(1\%)}$$

8. (10%) Let $f(x)$ be a differentiable and increasing function on $[a, b]$, where $a > 0$. Find a horizontal line $y = L$ that will minimize the function $F(L) = \int_a^b x|f(x) - L|\,\mathrm{d}x = \int_a^{f^{-1}(L)} x(L - f(x))\,\mathrm{d}x + \int_{f^{-1}(L)}^b x(f(x) - L)\,\mathrm{d}x$.

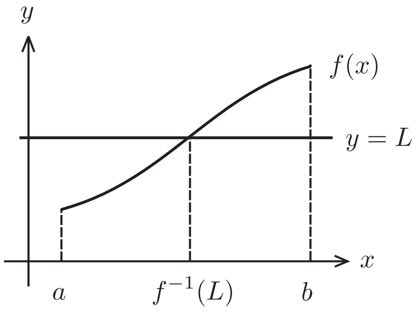


Figure 3: Find $y = L$ to minimize the function $F(L) = \int_a^b x|f(x) - L|\,\mathrm{d}x$.

Solution:

$$F(L) = L\int_a^{f^{-1}(L)} xdx - \int_a^{f^{-1}(L)} xf(x)dx + \int_{f^{-1}(L)}^b xf(x)dx - L\int_{f^{-1}(L)}^b xdx \text{ (1pt)}$$

$$\begin{aligned}
\frac{d}{dL}F(L) \text{ (1pt)} \quad &= \int_a^{f^{-1}(L)} xdx + L\cdot f^{-1}(L)\cdot \frac{d}{dL}f^{-1}(L) \\
&\quad - f^{-1}(L)\cdot L\cdot \frac{d}{dL}f^{-1}(L) - f^{-1}(L)\cdot L\cdot \frac{d}{dL}f^{-1}(L) \\
&\quad - \int_{f^{-1}(L)}^b xdx + f^{-1}(L)\cdot L\cdot \frac{d}{dL}f^{-1}(L) \\
&= \frac{1}{2}x^2\Big|_a^{f^{-1}(L)} - \frac{1}{2}x^2\Big|_{f^{-1}(L)}^b \\
&= \frac{1}{2}((f^{-1}(L))^2 - a^2) - \frac{1}{2}(b^2 - (f^{-1}(L))^2) = 0 \text{ (4pt)}
\end{aligned}$$

$$\Rightarrow\ (f^{-1}(L))^2 = \frac{1}{2}(a^2 + b^2) \text{ (2pt)}$$

$$\Rightarrow\ f^{-1}(L) = \pm\sqrt{\frac{a^2 + b^2}{2}} \text{ (choose } \sqrt{\frac{a^2 + b^2}{2}} \text{)}$$

$$\therefore f^{-1}(L) = \sqrt{\frac{a^2 + b^2}{2}} \text{ i.e. } L = f\left(\sqrt{\frac{a^2 + b^2}{2}}\right) \text{ (2pt)}$$

9. (12%)

(a) Suppose that $f(t)$ is piecewise continuous on $[0,\infty)$ and of exponential order. Let $F(s)=\mathfrak{L}\{f(t)\}$ be the Laplace transform of $f(t)$. Show that $\mathfrak{L}\{f(t-a)\mathcal{U}(t-a)\}=e^{-as}F(s)$, where $\mathcal{U}(t-a)$ is the unit step function defined as $\mathcal{U}(t-a)=\begin{cases}0, & \text{if } 0\le t<a\\ 1, & \text{if } t\ge a\end{cases}$.

(b) Express $g(t)=\begin{cases} t, & \text{if } 0\le t<1\\ 1, & \text{if } t\ge 1\end{cases}$ in terms of unit step functions.

(c) Solve the differential equation $y''+4y=g(t)$, where $y(0)=1$ and $y'(0)=0$.

Solution:

(a) (3 points) By the definition of the Laplace transform, we have

$$\begin{aligned}\mathcal{L}\{f(t-a)\mathcal{U}(t-a)\}&=\int_0^\infty \mathrm{e}^{-st}f(t-a)\mathcal{U}(t-a)\,\mathrm{d}t=\int_a^\infty \mathrm{e}^{-st}f(t-a)\,\mathrm{d}t\\&=\int_0^\infty \mathrm{e}^{-s(u+a)}f(u)\,\mathrm{d}u=\mathrm{e}^{-as}\int_0^\infty \mathrm{e}^{-st}f(t)\,\mathrm{d}t=\mathrm{e}^{-as}F(s),\end{aligned}$$

where $u=t-a, du=dt$.

• Partial credits: 1 point for the definition of the Laplace transform, 1 point for changing the variable.

(b) (2 points) $g(t)=t-(t-1)\mathcal{U}(t-1)$.

(c) (7 points) We take Laplace transform on both sides of the differential equation and get

$$\begin{aligned}s^2Y(s)-sy(0)-y'(0)+4Y(s)&=\frac{1}{s^2}-\mathrm{e}^{-s}\cdot\frac{1}{s^2}=\frac{1}{s^2}(1-\mathrm{e}^{-s})\\ \Rightarrow (s^2+4)Y(s)=s+\frac{1}{s^2}(1-\mathrm{e}^{-s})&\Rightarrow Y(s)=\frac{s}{s^2+4}+\frac{1}{s^2(s^2+4)}(1-\mathrm{e}^{-s}).\end{aligned}$$

Consider the partial fraction $\frac{1}{s^2(s^2+4)}=\frac{A}{s}+\frac{B}{s^2}+\frac{Cs+D}{s^2+4}$ for some constants $A,B,C,$ and D. We will solve $As(s^2+4)+B(s^2+4)+(Cs+D)s^2=1$, and it implies $A=0, B=\frac{1}{4}, C=0$, and $D=-\frac{1}{4}$. So we have

$$\begin{aligned}Y(s)&=\frac{s}{s^2+4}+\left(\frac{1}{4}\cdot\frac{1}{s^2}-\frac{1}{4}\cdot\frac{1}{s^2+4}\right)(1-\mathrm{e}^{-s})\\&=\frac{s}{s^2+2^2}+\left(\frac{1}{4}\cdot\frac{1}{s^2}-\frac{1}{8}\cdot\frac{2}{s^2+2^2}\right)(1-\mathrm{e}^{-s}).\end{aligned}$$

Hence the solution of the differential equation is

$$\begin{aligned}y(t)&=\cos(2t)+\frac{1}{4}t-\frac{1}{8}\sin(2t)-\left(\frac{1}{4}t\,\mathcal{U}(t)-\frac{1}{8}\sin(2t)\,\mathcal{U}(t)\right)\Big|_{t\to t-1}\\&=\cos(2t)+\frac{1}{4}t-\frac{1}{8}\sin(2t)-\left(\frac{1}{4}(t-1)\,\mathcal{U}(t-1)-\frac{1}{8}\sin(2(t-1))\,\mathcal{U}(t-1)\right).\end{aligned}$$

• Grading policy: 3 points for transforming the equation, 4 points for converting back to function in t: 1 point each for t, sine term, cosine term, and shifting operation.

9. (12%) Find the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the hemisphere $S = \{(x,y,z) | x^2 + y^2 + z^2 = 1, x \geq 0\}$.
Hint: The center of mass of an object is the average of the coordinates functions. You can consider the hemisphere as a surface of revolution obtained by rotating the curve $y = \sqrt{1-x^2}$, $x \geq 0$, about the x-axis.

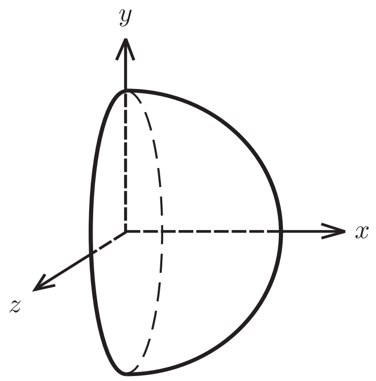


Figure 4: Find the center of mass $(\bar{x}, \bar{y}, \bar{z})$ of the hemisphere $S = \{(x,y,z) | x^2 + y^2 + z^2 = 1, x \geq 0\}$.

Solution:

By symmetry, $\bar{y} = \bar{z} = 0$. (2 points)

Viewing the hemisphere as a surface of revolution of a quarter circle $y = \sqrt{1-x^2}, x \geq 0$, about the x-axis, the average of the function x over the surface is

$$\bar{x} = \frac{\int_{x=0}^{x=1} x \cdot 2\pi y\, ds}{\int_{x=0}^{x=1} 2\pi y\, ds} \quad \text{(3 points)}$$

where $y = \sqrt{1-x^2}$, $\dfrac{dy}{dx} = \dfrac{-x}{\sqrt{1-x^2}}$, and $ds = \sqrt{(1 + \dfrac{dy}{dx})^2}\, dx = \dfrac{1}{\sqrt{1-x^2}} dx$ (3 points).

The denominator (2 points):

it is just the surface area, which is $\frac{1}{2}(4\pi \cdot 1^2) = 2\pi$ by known formula, or by integration

$$\int_{x=0}^{x=1} 2\pi y\, ds = 2\pi \int_0^1 \sqrt{1-x^2} \frac{1}{\sqrt{1-x^2}}\, dx = 2\pi.$$

The numerator (x-weighted surface; 2 points):

$$\int_{x=0}^{x=1} x \cdot 2\pi y\, ds = 2\pi \int_0^1 x \cdot \sqrt{1-x^2} \frac{1}{\sqrt{1-x^2}}\, dx = 2\pi \int_0^1 x\, dx = \pi.$$

Therefore, $\bar{x} = \dfrac{\pi}{2\pi} = \dfrac{1}{2}$, and the center of mass is located at $(\bar{x}, \bar{y}, \bar{z}) = (\frac{1}{2}, 0, 0)$.

(Note: if you misunderstood the problem and you correctly found the center of mass of the solid hemisphere $x^2 + y^2 + z^2 \leq 1, x \geq 0$ to be $(\frac{3}{8}, 0, 0)$, 8 points will be credited.)