1042微乙01-05班期中考解答和評分標準

1. (15%) 令 $f(x,y)=x^2e^{x\cos y}$ 其中 $x\in\mathbb{R}$, $-\frac{\pi}{2}<y<2\pi$。 求函數 $f(x)$ 的候選點並決定其極值性質。

Solution:

Let $f(x,\ y)=x^2e^{x\cos y}$, then $\nabla f(x,\ y)=(2xe^{x\cos y}+x^2e^{x\cos y}(\cos y),\ -x^2e^{x\cos y}(x\sin y))$.

$$\begin{cases} xe^{x\cos y}(2+x\cos y)=0 \text{ —(1)} \\ -x^3\sin y e^{x\cos y}=0 \text{ —(2)} \end{cases} \quad \text{(5 points)}$$

From (2), we have $x=0$ or $\sin y=0 \Rightarrow y=0$ or π.
(i) $x=0$, then $-\frac{\pi}{2}<y<2\pi$.
(ii) $y=0$, then $x=0$ or -2.
(iii) $y=\pi$, then $x=0$ or 2.
critical points: $(-2,\ 0)$, $(2,\ \pi)$, and $(0,\ y)$, $-\frac{\pi}{2}<y<2\pi$. (3 points)
$f_{xx}=e^{x\cos y}(2+4x\cos y+x^2\cos^2 y)$ (1 point)
$f_{xy}=f_{yx}=x^2e^{x\cos y}(-3\sin y-x\sin y\cos y)$ (1 point)
$f_{yy}=e^{x\cos y}(-x^3\cos y+x^4\sin^2 y)$ (1 point)
Let $D(x,\ y)=f_{xx}f_{yy}-(f_{xy})^2$.
(i) $D(-2,\ 0)<0$, then $(-2,\ 0)$ is a saddle point. (1 point)
(ii) $D(2,\ \pi)<0$, then $(2,\ \pi)$ is a saddle point. (1 point)
(iii) For $-\frac{\pi}{2}<y<2\pi$, $D(0,\ y)=0$, then it is inconclusion.
But we know that $f(x,\ y)\geq 0$ for all $x\in\mathbb{R}$, $-\frac{\pi}{2}<y<2\pi$. This implies that for all $-\frac{\pi}{2}<y<2\pi$, $f(0,\ y)=0$ is minimum. (2 points)

2. (10%) 求過曲面 $x^4+y^4+z^4=9xyz$ 上點 $(1,1,2)$ 的切平面方程式。

Solution:

Let $F(x,\ y,\ z)=x^4+y^4+z^4-9xyz$, then $\nabla F=(4x^3-9yz,\ 4y^3-9xz,\ 4z^3-9xy)$. (4 points)
$\Rightarrow \nabla F(1,\ 1,\ 2)=(-14,\ -14,\ 23)$ (3 points)
Then the tangent plane of $F=0$ at $(1,\ 1,\ 2)$ is $-14(x-1)-14(y-1)+23(z-2)=0$ or $-14x-14y+23z=18$. (3 points)

3. (15%) 限制條件 $xy=2$, 用 Lagrange 乘子法求函數 $\sqrt{x^2+4y^2}$ 的最小值。

Solution:

By Lagrange multiplier method

$$\begin{cases} 2x=\lambda y & (1) \\ 8y=\lambda x & (2) \\ xy=2 & (3) \end{cases} \quad \text{[5pts]}$$

by (3), if $\lambda=0$ then $x=y=0$ fail to satisfy (3) so $\lambda\neq 0$
Also by (3), x,y share the same sign
So λ must be positive [4pts]
multiple (1) and (2), one has $16xy=\lambda^2xy$ so $\lambda=4$ using (3) then $(x,y)=(\pm 2,\pm 1)$ [4pts, 2pts for each (x,y)]
the minimum value is $2\sqrt{2}$ [2pts]

4. (10%) 計算 $I=\int_{-1}^{1}\int_{-1}^{1}(e^{x^2}\sin y+x^2y^4)dxdy$.

Solution:

$$\begin{aligned}\int_{-1}^{1}\int_{-1}^{1}e^{x^2}\sin y\,dxdy &= \int_{-1}^{1}\int_{-1}^{1}e^{x^2}\sin y\,dydx \\ &= -\int_{-1}^{1}e^{x^2}\cos y\Big|_{-1}^{1}dx \\ &= -\int_{-1}^{1}e^{x^2}\big(\cos 1-\cos(-1)\big)dx \\ &= 0 \qquad \text{[5points]}\end{aligned}$$

$$\int_{-1}^{1}\int_{-1}^{1} x^2y^4dxdy = \int_{-1}^{1} \frac{1}{3}x^3y^4\Big|_{-1}^{1} dy$$
$$= \int_{-1}^{1} \frac{2}{3}y^4dy$$
$$= \frac{2}{15}y^5\Big|_{-1}^{1}$$
$$= \frac{4}{15} \qquad \text{[5points]}$$

$$\therefore I = \int_{-1}^{1}\int_{-1}^{1}(e^{x^2}\sin y + x^2y^4)dxdy = \int_{-1}^{1}\int_{-1}^{1} e^{x^2}\sin y dxdy + \int_{-1}^{1}\int_{-1}^{1} x^2y^4dxdy = \frac{4}{15}$$

5. (15%) 計算 $I = \iint_T (x+y)^{10}\,dxdy$, T 是一個三角形，三頂點座標為 $(0,0),(1,1),(2,0)$。

Solution:

We can do change of variables to find the integral. Set

$$\begin{cases} u = x+y \\ v = x-y \end{cases} \textit{(3 point)}$$

Then

$$\begin{cases} x = \frac{1}{2}(u+v) \\ y = \frac{1}{2}(u-v) \end{cases} \textit{(3 point)}$$

And the Jacobian will be

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2} \quad \textit{(3 point)}$$

So the integral will be

$$\iint_T (x+y)^{10}\,dx\,dy = \iint_{T'} u^{10}\left|-\frac{1}{2}\right|du\,dv$$

where T' is the triangle with vertices $(0,0)$, $(2,0)$, $(2,2)$.*(3 point)* And

$$\iint_{T'} \frac{1}{2}u^{10}\,du\,dv = \int_0^2\int_0^u \frac{1}{2}u^{10}\,dv\,du = \frac{512}{3} \quad \textit{(3 point)}$$

6. (10%) 計算 $I = \int_0^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2y^2dxdy$.

Solution:

Let $x = r\cos\theta$, $y = r\sin\theta$

$$\begin{aligned}
\text{原式} &= \int_0^{\pi}\int_0^2 r^4\cos^2\theta\sin^2\theta r dr d\theta \text{[2pts for integral range, 2pts for Jacobian determinant]} \\
&= \frac{2^6}{6}\int_0^{\pi} \frac{1+\cos 2\theta}{2}\frac{1-\cos 2\theta}{2}d\theta \text{ [2pts]} \\
&= \frac{2^4}{6}\int_0^{\pi} 1-\cos^2 2\theta d\theta \text{ [2pts]} \\
&= \frac{2^4}{6}\int_0^{\pi} 1-\frac{1+\cos 4\theta}{2}d\theta \text{ [2pts]} \\
&= \frac{4\pi}{3}
\end{aligned}$$

7. (10%) 計算 $I = \int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dxdy.$

Solution:

$$\begin{aligned}\int_0^1 \int_{\sqrt{y}}^1 e^{x^3}\,dxdy &= \int_0^1 \int_0^{x^2} e^{x^3}\,dydx \qquad \text{[4points]}\\ &= \int_0^1 ye^{x^3}\Big|_{y=0}^{y=x^2} dx\\ &= \int_0^1 x^2 e^{x^3}\,dx \qquad \text{[3points]}\\ &= \frac{1}{3}e^{x^3}\Big|_0^1\\ &= \frac{1}{3}(e-1) \qquad \text{[3points]}\end{aligned}$$

8. (15%) 令 $f(x,y) = \ln\sqrt{x^2+y^2}$.

(a) (5%) 求 f 在點 $(3,4)$ 沿著指向點 $(2,6)$ 的方向導數。

(b) (10%) 求 f 在點 $(3,4)$ 增加率最快的方向及沿此方向增加率的大小為何?

Solution:

(a)

$\nabla f(3,4) = (\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2})|_{(3,4)} = (\frac{3}{25}, \frac{4}{25})$ (2 points)

$\vec{u} = (2,6) - (3,4) = (-1,2)$ (1 points)

$\frac{\partial f}{\vec{u}}(3,4) = \partial f(3,4)\cdot\frac{\vec{u}}{|\vec{u}|} = (\frac{3}{25}, \frac{4}{25})\cdot(\frac{-1}{\sqrt{5}}, \frac{2}{\sqrt{5}}) = \frac{\sqrt{5}}{25}$ (2 points)

(b)

增加率最快方向= $\nabla f(3,4) = (\frac{3}{25}, \frac{4}{25})$ (5 points)

增加率= $|\nabla f(3,4)| = \sqrt{(\frac{3}{25})^2 + (\frac{4}{25})^2} = \frac{1}{5}$ (5 points)

如果$\nabla f(3,4)$計算錯誤而過程正確，可各得3分。