1. (10%) Evaluate the double integral $\int_1^2 \int_{\sqrt{x}}^{x} \frac{\sin y}{y} dydx + \int_2^4 \int_{\sqrt{x}}^{2} \frac{\sin y}{y} dydx$

**Solution:**
$= \int_1^2 \int_y^{y^2} \frac{\sin y}{y}\, dxdy$ (4pts)
$= \int_1^2 y \sin y dy - \int_1^2 \sin y dy$
$= -y \cdot \cos y|_{y=1}^2 + \int_1^2 \cos y dy + \cos 2 - \cos 1$
$= \sin 2 - \sin 1 - \cos 2$ (6pts)

2. (10%) Find the area of the region in the first quadrant enclosed by the curves $xy = a$, $xy = b$, $xy^{1.4} = c$ and $xy^{1.4} = d$ where $0 < a < b$ and $0 < c < d$.

**Solution:**
Let $R := \{(x, y) : a \le xy \le b, c \le xy^{1.4} \le d\}$,
then define $u = xy$ and $v = xy^{1.4} \Rightarrow x = \left(\frac{u^{1.4}}{v}\right)^{\frac{5}{2}}$ and $y = \left(\frac{v}{u}\right)^{\frac{5}{2}}$.
So we have $R' := \{(u, v) : a \le u \le b, c \le v \le d\}$,
and $|J| = |\begin{vmatrix} \frac{7}{2}u^{\frac{5}{2}}v^{-\frac{5}{2}} & -\frac{5}{2}u^{\frac{7}{2}}v^{-\frac{7}{2}} \\ -\frac{5}{2}u^{-\frac{7}{2}}v^{\frac{5}{2}} & \frac{5}{2}u^{-\frac{5}{2}}v^{\frac{3}{2}} \end{vmatrix}| = \left(\frac{35}{4} - \frac{25}{4}\right)v^{-1} = \frac{5}{2v}$
Area $= \iint_R 1 dA = \iint_{R'} 1 \times |J| dA' = \int_c^d \int_a^b \frac{5}{2v} dudv = \frac{5}{2}(b-a)\ln\frac{d}{c}$.

3. (10%) Find the mass of the solid $S$ bounded by the paraboloid $z = x^2 + 2y^2$ and the plane $z = 2 + 4y$ if $S$ has density function $\rho(x, y, z) = |x|$.

**Solution:**
There are two ways:

**Method 1:** The projection of $S$ onto $xy$-plane is the ellipse

$$D : \{(x, y) | x^2 + 2(y-1)^2 = 4\} \qquad (\mathbf{2\%})$$

Since $D$ is symmetric with respect to $yz$-plane, the mass of $S$ is twice of the part in the half space $x \ge 0$. Hence the mass is

$$\begin{aligned} m &= \iiint_S \rho\, dV = 2 \iiint_{S_x \ge 0} \rho\, dV \\ &= 2 \iint_{D_x \ge 0} \int_{x^2+2y^2}^{2+4y} x\, dzdA \qquad (\mathbf{3\%}) \\ &= 2 \int_{-\pi/2}^{\pi/2} \int_0^1 2r\cos\theta(4 - 4r^2) 2\sqrt{2}\, rdrd\theta \\ &= 32\sqrt{2} \int_{-\pi/2}^{\pi/2} \cos\theta \int_0^1 (r^2 - r^4)\, dr = \frac{128}{15}\sqrt{2} \qquad (\mathbf{2\%}) \end{aligned}$$

where we use the change of variables:

$$\begin{cases} x = 2r\cos\theta \\ y = 1 + \sqrt{2}r\sin\theta \end{cases}, \quad 0 \le r \le 1, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$$

with Jacobian

$$\begin{vmatrix} 2\cos\theta & -2r\sin\theta \\ \sqrt{2}r\sin\theta & \sqrt{2}r\cos\theta \end{vmatrix} = 2\sqrt{2} \qquad (\mathbf{3\%})$$

**Method 2:** The mass is given by

$$\begin{aligned} m &= 2\int_{1-\sqrt{2}}^{1+\sqrt{2}}\int_{2y^2}^{2+4y}\int_0^{\sqrt{z-2y^2}} x\ dxdzdy && (\mathbf{3\%}) \\ &= \int_{1-\sqrt{2}}^{1+\sqrt{2}}\int_{2y^2}^{2+4y}(z-2y^2)\ dzdy && (\mathbf{3\%}) \\ &= \int_{1-\sqrt{2}}^{1+\sqrt{2}} 2y^4-8y^3+4y^2+8y+2\ dy && (\mathbf{3\%}) \\ &= \frac{128}{15}\sqrt{2} && (\mathbf{1\%}) \end{aligned}$$

4. (10%) Let $S$ be a cone has radius $a$ and height $h$ without base. Evaluate the integral of the distance of the points to its axis over $S$.

**Solution:**

Solution 1. Under Cartesian coordinate system

$Z = \frac{a}{h}\sqrt{x^2+y^2}$

$\iint_S F\cdot dS = \iint_D F\cdot \mid r_x \times r_y \mid dA$

$r =< x,\ y,\ \frac{h}{a}\sqrt{x^2+y^2} >$

$r_x =< 1,\ 0,\ \frac{h}{a}\frac{x}{\sqrt{x^2+y^2}} >$

$r_y =< 0,\ 1,\ \frac{h}{a}\frac{y}{\sqrt{x^2+y^2}} >$

$\mid r_x \times r_y \mid = \sqrt{1+\frac{h^2}{a^2}}$

$\iint_D F\cdot \mid r_x \times r_y \mid dA = \iint_D \sqrt{x^2+y^2}\sqrt{1+\frac{h^2}{a^2}}\ dA$

Change the above equation from Cartesian coordinate to polar coordinate, and $0 \le r \le a$, $0 \le \theta \le 2\pi$ .

$\iint_D \sqrt{x^2+y^2}\sqrt{1+\frac{h^2}{a^2}}\ dA = \sqrt{1+\frac{h^2}{a^2}}\int_0^{2\pi}\int_0^a r^2\ drd\theta = \frac{2}{3}\pi a^2\sqrt{a^2+h^2}.$

Solution 2. Under cylindrical coordinate system

$r =< \rho\cos\theta,\ \rho\sin\theta,\ \frac{h}{a}\rho >$

$r_\rho =< \cos\theta,\ \sin\theta,\ \frac{h}{a} >$

$r_\phi =< -\rho\sin\theta,\ \rho\cos\theta,\ 0 >$

$\mid r_\rho \times r_\phi \mid = \rho\sqrt{1+\frac{h^2}{a^2}}$

Therefore, $\int_0^{2\pi}\int_0^{\sqrt{a^2+h^2}} \rho^2\ d\rho d\phi = \frac{2}{3}\pi a^2\sqrt{a^2+h^2}$

Solution 3. Under Spherical coordinate system

Let $\tan\alpha = \frac{a}{h} \Rightarrow \alpha = \tan^{-1}\frac{a}{h}$.

$r =< \rho\cos\theta\sin\alpha,\ \rho\sin\theta\sin\alpha,\ \cos\alpha >$

$\mid r_\rho \times r_\phi \mid = \rho\sin\alpha$

$$\int_0^{2\pi}\int_0^{\sqrt{a^2+h^2}} \rho\sin\alpha\ \rho\sin\alpha\ d\rho\ d\theta = \int_0^{2\pi}\sin\alpha\int_0^{\sqrt{a^2+h^2}} \rho^2\ d\rho = \frac{2}{3}\pi a^2\sqrt{a^2+h^2}$$

Grading policies:
(A) 2 points for the corret formula for parametric surface.
(B) Another 3 points were given when the Jacobian term is correct.
(C) The other 2 points were given when integrate formula and regions are correct.
(D) The last 3 points were given when integration is correct.

5. (18%) Consider the vector field defined by $\mathbf{G}(x,y) = (3x^2+y)\mathbf{i} + (2x^2y - x)\mathbf{j}$, $(x,y) \in \mathbb{R}^2$.
   (a) Is $\mathbf{G}(x,y)$ conservative?
   (b) Find a function $\mu(x)$ with $\mu(1) = 1$ such that $\mu(x)\mathbf{G}(x,y)$ is conservative.
   (c) Set $\mathbf{F}(x,y)$ to be the conservative vector field in (b). Find the potential function $f(x,y)$ of $\mathbf{F}$ with $f(1,0) = 3$.
   (d) Let $C$ be the curve with defining equation in polar coordinate given by

$$r = \sec\theta + \frac{\sqrt{2}}{\pi}\theta,\ \theta \in \left[0, \frac{\pi}{4}\right].$$

   Evaluate the integral $\int_C \mathbf{F}\cdot d\mathbf{r}$.

**Solution:**

(a) Let $P = 3x^2 + y$, $Q = 2x^2y - x$. Since $\dfrac{\partial Q}{\partial x} = 4xy - 1$, $\dfrac{\partial P}{\partial y} = 1$
$\therefore \mathbf{G}(x,y)$ is not conservative (2pts)

(b) $\dfrac{\partial}{\partial x}(\mu(x)\cdot Q) = \dfrac{\partial}{\partial y}(\mu(x)\cdot P) = \mu(x)\cdot 1$
$\Rightarrow \mu'(x)(2x^2y - x) + \mu(x)(4xy-1) = \mu(x)$
$\Rightarrow \mu'(x)(2x^2y - x) = \mu(x)(2 - 4xy)$
$\Rightarrow \dfrac{\mu'(x)}{\mu(x)} = \dfrac{2(1-2xy)}{x(2xy-1)} = \dfrac{-2}{x}$ (5pts)
$\Rightarrow \ln|\mu(x)| = -2\ln x + C_0$
$\Rightarrow |\mu(x)| = C_1 x^{-2}$ (2pts)
Since $\mu(1) = 1$, then $\mu(x) = x^{-2}$ (1pt)

(c) $\mu(x)\mathbf{G}(x) = (3 + \dfrac{y}{x^2})\mathbf{i} + (2y - \dfrac{1}{x})\mathbf{j} = \triangledown f(x,y)$ (1pt)
$\Rightarrow f(x,y) = 3x - \dfrac{y}{x} + y^2 + C$ (2pts)
$\Rightarrow f(1,0) = 3 + C = 3 \Rightarrow C = 0$
$\Rightarrow f(x,y) = 3x - \dfrac{y}{x} + y^2$ (1pt)

(d) $r(\dfrac{\pi}{4}) = (\sqrt{2} + \dfrac{\sqrt{2}}{4})\cdot(\cos\dfrac{\pi}{4}, \sin\dfrac{\pi}{4}) = (\dfrac{5}{4}, \dfrac{5}{4})$ (1pt)
$r(0) = (1+0)\cdot(\cos 0, \sin 0) = (1,0)$ (1pt)
$\int_C \mathbf{F}\cdot dr = f(\dfrac{5}{4}, \dfrac{5}{4}) - f(1,0) = \dfrac{15}{4} - 1 + \dfrac{25}{16} - 3 = \dfrac{21}{16}$ (2pts)

6. (12%) Let $\mathbf{F}(x,y) = \dfrac{y^3}{(x^2+y^2)^2}\mathbf{i} - \dfrac{xy^2}{(x^2+y^2)^2}\mathbf{j}$.
   (a) Show that $\mathbf{F}$ is conservative on the domain $D = \mathbb{R}^2 - \{(0,y)|y \le 0\}$.
   (b) Compute $\int_C \mathbf{F}\cdot d\mathbf{r}$, where $C$ is the part of the polar curve $r = 1 + \sin\theta$, $0 \le \theta \le \pi$.

**Solution:**

(a) For $P(x,y) = \dfrac{y^3}{(x^2+y^2)^2}$ and $Q(x,y) = \dfrac{-xy^2}{(x^2+y^2)^2}$, we have

$$\frac{\partial P}{\partial y} = \frac{3x^2y^2 - y^4}{(x^2+y^2)^3} = \frac{\partial Q}{\partial x}$$ (**5%**)

Since the domain $D$ is simply connected (1%), $F$ has a potential on $D$ and is conservative.

(b) There are three ways:

**Method 1:** Since $F$ is conservative on $D$, for any curve $C'$ in $D$ with the same starting point and end point of $C$, we have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C'} \mathbf{F} \cdot d\mathbf{r} \qquad (3\%)$$

Take $C'$ to be the upper half unit circle $(\cos t, \sin t)$, $0 \le t \le \pi$, then

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_{C'} \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^{\pi} (\sin t)^3(-\sin t) - (\cos t \sin^2 t)(\cos t)\ dt && (2\%) \\ &= \int_0^{\pi} -\sin^2 t dt = -\frac{\pi}{2} && (1\%)\end{aligned}$$

**Method 2:** We find the potential function $f$ with $\nabla f = F$.

$$\begin{aligned}\int \frac{y^3}{(x^2+y^2)^2} dx &= \int \frac{-xy^2}{(x^2+y^2)} dy \\ &= y^3 \int \frac{y \sec\theta}{(y\sec\theta)^2} d\theta \qquad (x = y\tan\theta) \\ &= \int \frac{1}{\sec^2\theta} d\theta = \frac{1}{2}\theta + \frac{\sin 2\theta}{4} \\ &= \frac{1}{2}\tan^{-1}(\frac{x}{y}) + \frac{1}{2}\cdot\frac{xy}{x^2+y^2} && (5\%)\end{aligned}$$

It is easy to see that the function

$$f(x,y) = \frac{1}{2}\tan^{-1}(\frac{x}{y}) + \frac{1}{2}\cdot\frac{xy}{x^2+y^2}$$

satisfies $f_y(x,y) = Q(x,y)$ and hence is a potential function. Hence

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(-1,0) - f(1,0) = \frac{1}{2}(-\frac{\pi}{2} - \frac{\pi}{2}) = -\frac{\pi}{2} \qquad (1\%)$$

*Remark* 0.1. A potential function of $\mathbf{F}$ on $D$ is given by

$$\tilde{f}(x,y) = \begin{cases} f(x,y) = f_+(x,y), & (x,y) \in I \\ f(x,y) = f_-(x,y), & (x,y) \in II \\ f_-(x,y), & (x,y) \in III \\ f_+(x,y), & (x,y) \in IV \end{cases}$$

where

$$\begin{cases} f_+(x,y) = \frac{1}{2}(\frac{\pi}{2} - \tan^{-1}(\frac{y}{x})) + \frac{1}{2}\cdot\frac{xy}{x^2+y^2}, & x > 0 \\ f_-(x,y) = \frac{1}{2}(-\frac{\pi}{2} - \tan^{-1}(\frac{y}{x})) + \frac{1}{2}\cdot\frac{xy}{x^2+y^2}, & x < 0 \end{cases}$$

**Method 3:** A parametrization of the curve $C$ is

$$\gamma(\theta) = ((1+\sin\theta)\cos\theta, (1+\sin\theta)\sin\theta),\ 0 \le \theta \le \pi,$$

and vector field $\mathbf{F}$,

$$\mathbf{F}(\theta) = \left(\frac{((1+\sin\theta)\sin\theta)^3}{(1+\sin\theta)^4}, \frac{-(1+\sin\theta)^3\sin^2\theta\cos\theta}{(1+\sin\theta)^4}\right) \qquad (2\%)$$

Since

$$\gamma'(\theta) = (\cos^2\theta - (1+\sin\theta)\sin\theta, \cos\theta\sin\theta + (1+\sin\theta)\cos\theta) \qquad (\mathbf{2\%})$$

the line integral is

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^\pi -\sin^4\theta - \sin^2\theta\cos^2\theta \; d\theta \qquad (\mathbf{1\%})$$

$$= \int_0^\pi -\sin^2\theta \; d\theta = -\frac{\pi}{2} \qquad (\mathbf{1\%})$$

7. (10%) Let $C$ be the curve formed by the intersection of the plane $z = x$ and the surface $z = x^2 + y^2$. $C$ is oriented counterclockwise when viewed from above. Evaluate $\oint_C (xyz + \tan^{-1} x)dx + (x^2 + \sinh y)dy + (xz + \ln z)dz$.

**Solution:**

By Stoke's Theorem, $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \text{curl}\mathbf{F}\cdot d\mathbf{S} = \iint_S \nabla\times\mathbf{F}\cdot d\mathbf{S}$.

(Method I) Let $S_1 := \{(x,y,z) : z \geq x^2 + y^2, z = x\}$.

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xyz + \tan^{-1} x & x^2 + \sinh y & xz + \ln z \end{vmatrix} = (0-0)\,\mathbf{i} + (xy - z)\,\mathbf{j} + (2x - xz)\,\mathbf{k}$$

$$\iint_{S_1} \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_{S_1'} (0, xy - z, 2x - xz)\cdot(-1,0,1)dA = \iint_{S_1'} 2x - x^2 dA,$$

where $S_1' = \{(x,y) : x^2 - x + y^2 \leq 0\}$.

I - 1 : Then by polar coordinate $x^2 - x + y^2 = 0 \Rightarrow r = \cos\theta$

$\Rightarrow S_1'$ is the region enclosed by the curve $r = \cos\theta$.

$$\Rightarrow \int_0^\pi \int_0^{\cos\theta} \left(2r\cos\theta - r^2\cos^2\theta\right) r drd\theta = \int_0^\pi \frac{2}{3}r^3\cos\theta - \frac{1}{4}r^4\cos^2\theta\bigg|_0^{\cos\theta} d\theta$$

$$= \int_0^\pi \frac{2}{3}\cos^4\theta - \frac{1}{4}\cos^6\theta d\theta = \int_0^\pi \frac{2}{3}\left(\frac{1+\cos 2\theta}{2}\right)^2 - \frac{1}{4}\left(\frac{1+\cos 2\theta}{2}\right)^3 d\theta$$

$$= \int_0^\pi \frac{1}{6}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right) - \frac{1}{32}\left(1 + 3\cos 2\theta + 3\cos^2 2\theta + \cos^3 2\theta\right) d\theta$$

$$= \left(\frac{1}{6} - \frac{1}{32}\right)\pi + \left(\frac{1}{6} - \frac{3}{32}\right)\int_0^\pi \cos^2 2\theta d\theta = \frac{13}{96}\pi + \frac{25}{96}\int_0^\pi \frac{1+\cos 4\theta}{2} d\theta$$

$$= \frac{13}{96}\pi + \frac{7}{96}\cdot\frac{1}{2}\pi = \frac{33}{192}\pi = \frac{11}{64}\pi$$

I - 2 : Let $\begin{cases} x = r\cos\theta + \dfrac{1}{2} \\ y = r\sin\theta \end{cases} \Rightarrow \iint_{S_1'} 2x - x^2 dA = \int_0^{2\pi}\int_0^{\frac{1}{2}} \left(2\left(r\cos\theta + \frac{1}{2}\right) - \left(r\cos\theta + \frac{1}{2}\right)^2\right) rdrd\theta$

$$= \int_0^{2\pi}\int_0^{\frac{1}{2}} 2r\cos\theta + 1 - \left(r^2\cos^2\theta + r\cos\theta + \frac{1}{4}\right) rdrd\theta = \int_0^{2\pi}\int_0^{\frac{1}{2}} \frac{3}{4}r - r^3\cos^2\theta drd\theta$$

$$= 2\pi\cdot\frac{3}{8}\cdot\left(\frac{1}{2}\right)^2 - \frac{1}{4}\left(\frac{1}{2}\right)^4 \int_0^{2\pi} \frac{1+\cos 2\theta}{2} d\theta = \frac{6\pi}{32} - \frac{\pi}{64} = \frac{11}{64}\pi$$

(Method II) Let $S_2 := \{(x,y,z) : z = x^2 + y^2, z \geq x\}$,

$\Rightarrow \mathbf{r}(u,v) = \left(u, v, u^2 + v^2\right) \Rightarrow \mathbf{r}_u \times \mathbf{r}_v = (1,0,2u)\times(0,1,2v) = (-2u, -2v, 1)$

$$\iint_{S_2} \nabla\times\mathbf{F}\cdot d\mathbf{S} = \iint_{S_2'} \left(0, uv - \left(u^2+v^2\right), 2u - u\left(u^2+v^2\right)\right)\cdot(-2u,-2v,1)dA$$

$$= \iint_{S_2'} -2uv^2 + 2v\left(u^2+v^2\right) + 2u - u\left(u^2+v^2\right) dA = \iint_{S_2'} 2v^3 - 3uv^2 + 2u^2v + 2u - u^3 dA$$

$$= \iint_{S_2'} -3uv^2 + 2u - u^3,$$

where $S_2' = \{(u,v) : u^2 - u + v^2 \leq 0\}$.

II - 1 : Then by polar coordinate $u^2 - u + v^2 = 0 \Rightarrow r = \cos\theta$

$\Rightarrow S_2'$ is the region enclosed by the curve $r = \cos\theta$.

$$\Rightarrow \int_0^{\pi} \int_0^{\cos\theta} \left(-3r^3 \cos\theta \sin^2\theta + 2r\cos\theta - r^3\cos^3\theta\right) r dr d\theta$$

$$= \int_0^{\pi} -\frac{3}{5}r^5 \cos\theta\sin^2\theta + \frac{2}{3}r^3\cos\theta - \frac{1}{5}r^5\cos^3\theta\Big|_0^{\cos\theta} d\theta$$

$$= \int_0^{\pi} -\frac{3}{5}\cos^6\theta\sin^2\theta + \frac{2}{3}\cos^4\theta - \frac{1}{5}\cos^8\theta d\theta$$

$$= \int_0^{\pi} -\frac{3}{5}\left(\frac{1+\cos 2\theta}{2}\right)^3 + \frac{2}{3}\left(\frac{1+\cos 2\theta}{2}\right)^2 + \frac{2}{5}\left(\frac{1+\cos 2\theta}{2}\right)^4 d\theta$$

$$= \int_0^{\pi} \frac{1}{6}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right) - \frac{3}{40}\left(1 + 3\cos 2\theta + 3\cos^2 2\theta + \cos^3 2\theta\right)$$

$$+ \frac{1}{40}\left(1 + 4\cos 2\theta + 6\cos^2 2\theta + 4\cos^3 2\theta + \cos^4 2\theta\right) d\theta$$

$$= \left(\frac{1}{6} - \frac{3}{40} + \frac{1}{40}\right)\pi + \left(\frac{1}{6} - \frac{9}{40} + \frac{6}{40}\right)\int_0^{\pi} \cos^2 2\theta d\theta + \frac{1}{40}\int_0^{\pi} \cos^4 2\theta d\theta$$

$$= \frac{7}{60}\pi + \frac{11}{120}\int_0^{\pi} \frac{1+\cos 4\theta}{2} d\theta + \frac{1}{40}\int_0^{\pi}\left(\frac{1+\cos 4\theta}{2}\right)^2 d\theta$$

$$= \frac{7}{60}\pi + \frac{11}{120}\cdot\frac{1}{2}\pi + \frac{1}{160}\int_0^{\pi} 1 + 2\cos 4\theta + \cos^2 4\theta d\theta$$

$$= \frac{39}{240} + \frac{1}{160}\pi + \frac{1}{160}\int_0^{\pi}\frac{1+\cos 8\theta}{2}d\theta = \frac{27}{160}\pi + \frac{1}{160}\cdot\frac{1}{2}\pi = \frac{55}{320}\pi = \frac{11}{64}\pi$$

II - 2 : Let $\begin{cases} u = r\cos\theta + \dfrac{1}{2} \\ v = r\sin\theta \end{cases} \Rightarrow \iint_{S_2'} 2v^3 - 3uv^2 + 2u^2v + 2u - u^3 dA$

$$= \int_0^{2\pi}\int_0^{\frac{1}{2}} \left(-3\left(r\cos\theta + \frac{1}{2}\right)r^2\sin^2\theta + 2\left(r\cos\theta + \frac{1}{2}\right) - \left(r\cos\theta + \frac{1}{2}\right)^3\right) r dr d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{1}{2}} -3r^4\cos\theta\sin^2\theta + \frac{7}{8}r - \frac{3}{2}r^3 dr d\theta$$

$$= \int_0^{2\pi} \frac{7}{16}\cdot\frac{1}{4} - \frac{3}{8}\cdot\frac{1}{16} d\theta = \frac{11}{64}\pi$$

8. (10%) Let $S$ be the surface $x^2 + y^2 + z^2 = 1$, $x, y, z \geq 0$, an eighth of a sphere, and $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$. Find the outward flux of $\mathbf{F}$ across $S$.

**Solution:**

Method 1:

Calculate flux directly.

$\iint F \cdot n\, ds = \iint (x^2, y^2, z^2)\cdot(x, y, z)\, ds = \iint x^3 + y^3 + z^3\, ds = 3\iint z^3\, ds$ By symmetry.

$\iint z^3\, ds = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos^3\phi\sin\phi\, d\phi d\theta$ (5pts)

$= \frac{\pi}{2}(-\frac{\cos^4\phi}{4}|_0^{\frac{\pi}{2}}) = \frac{\pi}{8}$ (5pts)

Thus $\iint F\cdot n\, ds = 3\iint z^3\, ds = \frac{3\pi}{8}$

Method 2:

Use Divergence Theorem.

$\iint_S F\cdot n\, ds + \iint_{S_1\cup S_2\cup S_3} F\cdot n\, ds = \iiint divF\, dE$ (2pts)

where $S_1$ is $\{(x, y, z) \in \mathbb{R}^3 | x = 0, y \geq 0, z \geq 0, y^2 + z^2 \leq 1\}$

$S_2$ and $S_3$ have the same shape as $S_1$ but in $y = 0$ and $z = 0$ .

First, take a look at $\iint_{S_1} F\cdot n\, ds$, $F = (0, y^2, z^2)$ while the normal vector is $(-1, 0, 0)$

We have $\iint_{S_1} F\cdot n\, ds = 0$.

For similar reason $\iint_{S_2} F\cdot n\, ds = \iint_{S_3} F\cdot n\, ds = 0$ (3pts)

Now we have $\iint_S F\cdot n\, ds = \iiint divF\, dE = \iiint 2x + 2y + 2z\, dE = 6\iiint z\, dE$ by symmetry.

$\iiint z\,dE = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \int_0^1 \rho\cos\phi \cdot \rho^2 \sin\phi \, d\rho d\phi d\theta$ (2pts)
$= \frac{1}{4} \cdot \frac{\pi}{2} \cdot \frac{\sin^2\phi}{2}\Big|_{\phi=0}^{\frac{\pi}{2}} = \frac{\pi}{16}$ (3pts)
Thus we have $\iint_S F \cdot n\,ds = 6 \iiint z\,dE = \frac{3\pi}{8}$

9. (10%) Solve the initial value problem

$$\begin{cases} y'' + y = xe^x + \sec x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ y(0) = 1, \quad y'(0) = -3. \end{cases}$$

**Solution:**
The auxiliary equation is $r^2 + 1 = 0$, whose roots are $\pm i$, so the solution of $y'' + y = 0$ is

$$y_c(x) = C_1 \cos x + C_2 \sin x \quad \textbf{(2pts)}$$

For a particular solution of $y'' + y = xe^x$, we try $y_{p_1}(x) = (Ax + B)e^x$.
Then $y''_{p_1} + y_{p_1} = \Big(2Ax + (2A + 2B)\Big)e^x$, and then we have

$$\begin{cases} A = \frac{1}{2} \\ B = -\frac{1}{2} \end{cases}$$

Thus a particular solution is

$$y_{p_1}(x) = \frac{1}{2}(x-1)e^x \quad \textbf{(3pts)}$$

For a particular solution of $y'' + y = \sec x$, we use variation of parameters to seek a solution of the form $y_{p_2}(x) = u_1(x)\cos x + u_2(x)\sin x$.
If we set $u_1' \cos x + u_2' \sin x = 0$, then $u_1'\Big(\frac{d}{dx}\cos x\Big) + u_2'\Big(\frac{d}{dx}\sin x\Big) = \sec x$.

$$\Rightarrow \begin{cases} u_1' = -\tan x \\ u_2' = 1 \end{cases}$$

$$\Rightarrow \begin{cases} u_1 = \ln(\cos x) \\ u_2 = x \end{cases}$$

Then we obtain

$$y_{p_2}(x) = \cos x \ln(\cos x) + x \sin x \quad \textbf{(3pts)}$$

Therefore, the general solution of $y'' + y = xe^x + \sec x$ is $y(x) = y_c(x) + y_{p_1}(x) + y_{p_2}(x)$.

$$\Rightarrow y(x) = C_1 \cos x + C_2 \sin x + \frac{1}{2}(x-1)e^x + \cos x \ln(\cos x) + x \sin x$$

To satisfy the initial conditions we require that

$$\begin{cases} y(0) = C_1 - \frac{1}{2} = 1 \\ y'(0) = C_2 = -3 \end{cases}$$

$$\Rightarrow \quad \begin{cases} C_1 = \frac{3}{2} \\ C_2 = -3 \end{cases}$$

Thus the required solution of the initial-value problem is

$$y(x) = \frac{3}{2}\cos x - 3\sin x + \frac{1}{2}(x-1)e^x + \cos x \ln(\cos x) + x \sin x \quad \textbf{(2pts)}$$