1. (8%) Find the limit

$$\lim_{x\to 0} \frac{\tan^{-1}(\sin(ax))}{\tan(\sin^{-1}(bx))},$$

where a and b are constants, and $b \neq 0$.

Solution:

By L'Hôpital Rule and Chain Rule, we have

$$\lim_{x\to 0} \frac{\arctan(\sin(ax))}{\tan(\arcsin(bx))} = \lim_{x\to 0} \frac{\frac{1}{1+\sin^2(ax)}\cos(ax)a}{\sec^2(\arcsin(bx))\frac{1}{\sqrt{1-(bx)^2}}b} = \frac{a}{b}$$

[Method 2]
By trigonometric function, we know

$$\tan(\arcsin(bx)) = \frac{bx}{\sqrt{1-(bx)^2}}$$

So we get

$$= \lim_{x\to 0} \frac{\arctan(\sin(ax))}{\frac{bx}{\sqrt{1-(bx)^2}}} = \lim_{x\to 0} \frac{\frac{1}{1+\sin^2(ax)}\cos(ax)a}{\frac{b\sqrt{1-(bx)^2} - \frac{b^3x^2}{\sqrt{1-(bx)^2}}}{1-(bx)^2}} = \frac{a}{b}$$

[Method 3]

$$= \lim_{x\to 0} \frac{\arctan(\sin(ax))}{\sin(ax)} \frac{\arcsin(bx)}{\tan(\arcsin(bx)} \frac{\sin(ax)}{ax} \frac{bx}{arcsin(bx)} \frac{ax}{bx} = \frac{a}{b}$$

Policy:
If you completely solve this problem, you can get 8 points.
If you use L'Hôpital Rule but you don't use Chain Rule, you will lose 1 point to 4 points.
If you use L'Hôpital Rule wrongly, you will lose 1 point to 3 points.

2. (10%) Figure 1 shows a circle with radius 1 inscribed in the parabola $y = 2x^2$. Find the center of the circle C, and find the tangent points P and Q.

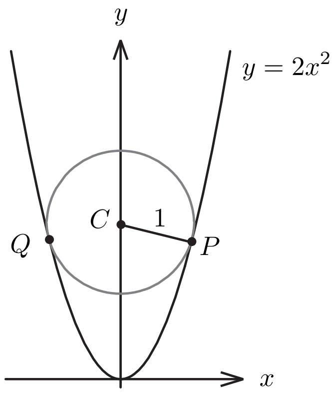


Figure 1: A unit circle is inscribed in the parabola $y = 2x^2$.

Solution:

First, we set $C(0, c), P(a, 2a^2), Q(-a, 2a^2)$, where $a > 0$. And we can know tangent line of P is $y - 2a^2 = 4a(x - a)$. And we know two prependicular slope m_1 and m_2 satisfying $m_1 m_2 = -1$ So

$$\frac{2a^2 - c}{a - 0} \times 4a = -1$$

. So we have

$$2a^2 - c = -\frac{1}{4}$$

On the other hand, because the radius of the circle is 1. So we have

$$a^2 + (2a^2 - c)^2 = 1$$

so we have

$$a^2 + (\frac{1}{4})^2 = 1$$

so we get

$$a = \frac{\sqrt{15}}{4}$$

Therefore we can know

$$c = 2a^2 + \frac{1}{4} = 2 \times (\frac{\sqrt{15}}{4})^2 + \frac{1}{4} = \frac{17}{8}$$

So $C(0, \frac{17}{8}), P(\frac{\sqrt{15}}{4}, \frac{15}{8}), Q(-\frac{\sqrt{15}}{4}, \frac{15}{8})$

Policy:
If you completely solve this problem, you can get 10 points.
If your answer is correct, but you have no get C or P or Q, you will lose 1 point to 2 points.
If you have calculation error, you will lose 1 to 5 points.
If you have no or have less idea to deal this problem, you will get 0 point to 4 points.

3. (10%) Student A used some mathematical software to plot a dolphin-like curve as Figure 2.

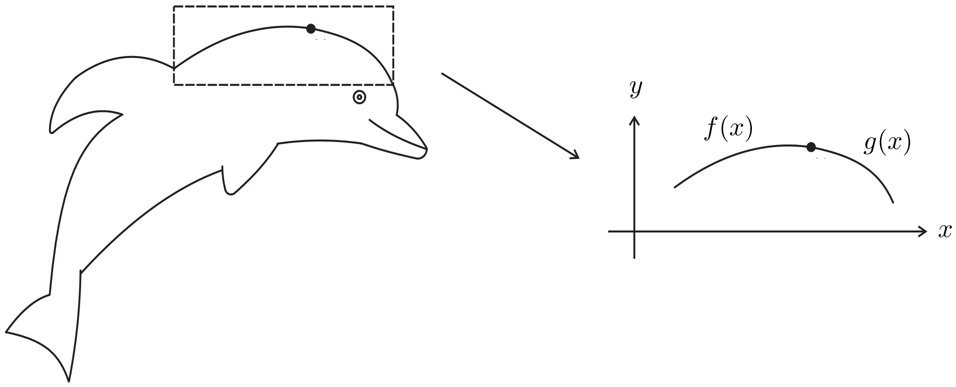


Figure 2: Dolphin-like curve.

On the dolphin's back, he considered two functions:

$$f(x) = -\frac{1}{8}\left(x - \frac{7}{2}\right)^2 + \frac{65}{32}, \quad 0.7 < x \le 4$$
$$g(x) = \frac{a}{x-7} + b, \qquad 4 < x < 6.3.$$

Find constants a and b such that the union of two functions $f(x)$ and $g(x)$ is differentiable on $(0.7, 6.3)$.

Solution:

$f(4) = \frac{-1}{8}\left(4 - \frac{7}{2}\right)^2 + \frac{65}{32} = 2$ (1 point)

$g(4) := \lim_{x\to 4^+} g(x) = -\frac{1}{3}a + b$ (1 point)

$f(4) = g(4) \Rightarrow -\frac{1}{3}a + b = 2$ (Because we want $f(x)$ and $g(x)$ to be continuous, we set $f(4) = g(4)$.) (2 points)

$\lim_{\delta\to 0^+} \frac{f(4-\delta) - f(4)}{-\delta} = \frac{-1}{8}$ (1 point)

$\lim_{\delta\to 0^+} \frac{g(4+\delta) - g(4)}{\delta} = \frac{-1}{9}a$ (1 point)

$\frac{-1}{8} = \frac{-1}{9}a$ (Because we want $f(x)$ and $g(x)$ to be differentiable, we set that left limit equal to right limit.) (2 points)

We solve the simultaneous equations, and we can get $a = \frac{9}{8}$, $b = \frac{19}{8}$. (2 points)

4. (10%) A rectangle has vertices $(-x,0),(x,0),(x,y),(-x,y)$, where $y \geq 0$ and where $x^2+y^2=1$. Suppose that x is changing with the time t in the way $x(t)=t^2, -1<t<1$.

   (a) (5%) Find the rate of change of $y(t)$ with respect to t.

   (b) (5%) Find the rate of change of the area of the rectangle with respect to t.

Solution:

(a)
method1:
$x^2+y^2=1$ & $x(t)=t^2$
$\Rightarrow y(t)=\sqrt{1-t^4}$ (2%)
$\Rightarrow y'(t)=\dfrac{-2t^3}{\sqrt{1-t^4}}$ (3%)

method2:
$x^2+y^2=1$
$\Rightarrow 2x\dfrac{dx}{dt}+2y\dfrac{dy}{dt}=0$ (3%)
substitute $x(t)=t^2,\ x'(t)=2t,\ y(t)=\sqrt{1-t^4}$
$\Rightarrow y'(t)=\dfrac{-2t^3}{\sqrt{1-t^4}}$ (2%)

(b)
method1:
area $A(t)=2x(t)y(t)=2t^2\sqrt{1-t^4}$ (2%)
$\Rightarrow A'(t)=4t\sqrt{1-t^4}+\dfrac{-4t^5}{\sqrt{1-t^4}}=\dfrac{4t-8t^5}{\sqrt{1-t^4}}$ (3%)

method2:
area $A(t)=2x(t)y(t)$
$\Rightarrow A'(t)=2\dfrac{dx}{dt}y+2x\dfrac{dy}{dt}$ (3%)
substitute $x(t)=t^2,\ x'(t)=2t,\ y(t)=\sqrt{1-t^4},\ y'(t)=\dfrac{-2t^3}{\sqrt{1-t^4}}$
$\Rightarrow A'(t)=4t\sqrt{1-t^4}+\dfrac{-4t^5}{\sqrt{1-t^4}}=\dfrac{4t-8t^5}{\sqrt{1-t^4}}$ (2%)

5. Let $f(x) = \dfrac{-x^2+3x-1}{x^2+1}$. Answer the following questions by filling each blank below. Show your work (computations and reasoning) in the space following. Put **None** in the blank if the item asked does **not** exist.

(a) The function is increasing on the interval(s) ____________________ and decreasing on the interval(s) ____________________. (6% total)

The local maximal point(s) $(x, y) =$ ____________________. (2%)

The local minimal point(s) $(x, y) =$ ____________________. (2%)

Reason:

Solution:

$$f'(x) = \frac{(-2x+3)(x^2+1)-(-x^2+3x-1)2x}{(x^2+1)^2} = \frac{-3x^2+3}{(x^2+1)^2}$$

(2 points。式子最後化簡錯扣1分)

$f'(x) = 0 \Longleftrightarrow x = \pm 1$

	$x<-1$	$-1<x<1$	$x>1$
f'	$-$	$+$	$-$
f	$\searrow$	$\nearrow$	$\searrow$

f is increasing on $(-1, 1)$ (2 points)

f is decreasing on $(-\infty, -1) \cup (1,\infty)$ (2 points)

f has local maximum $(1, f(1)) = (1, \frac{1}{2})$ (2 points)

f has local minimum $(1, f(-1)) = (-1, \frac{-5}{2})$ (2 points)

(b) The function is concave upward on the interval(s) ____________________ and concave downward on the interval(s) ____________________. (4% total)

The inflection point(s) $(x, y) =$ ____________________. (3%)

Reason:

Solution:

$$\begin{aligned} f''(x) &= -3 \times \frac{2x(x^2+1)^2-(x^2-1)2(x^2+1)2x}{(x^2+1)^4} \\ &= -3 \times \frac{2x(-x^2+3)}{(x^2+1)^3} = \frac{6x(x^2-3)}{(x^2+1)^3} \end{aligned}$$

(2 points。式子最後化簡錯扣1分)

$f''(x) = 0 \Leftrightarrow x = 0, \pm\sqrt{3}$

	$x<-\sqrt{3}$	$-\sqrt{3}<x<0$	$0<x<\sqrt{3}$	$x>\sqrt{3}$
f''	$-$	$+$	$-$	$+$

f concaves up on $(-\sqrt{3}, 0) \cup (\sqrt{3}, \infty)$ (1 point)

f concaves down on $(-\infty, -\sqrt{3}) \cup (0, \sqrt{3})$ (1 point)

inflection points $(0, 1), (-\sqrt{3}, \frac{-3\sqrt{3}-4}{4}), (\sqrt{3}, \frac{3\sqrt{3}-4}{4})$ (3 points)

(c) The vertical asymptotes lines of the function are ____________________.

The horizontal asymptotes lines are ____________________. (3% total)

Reason:

Solution:

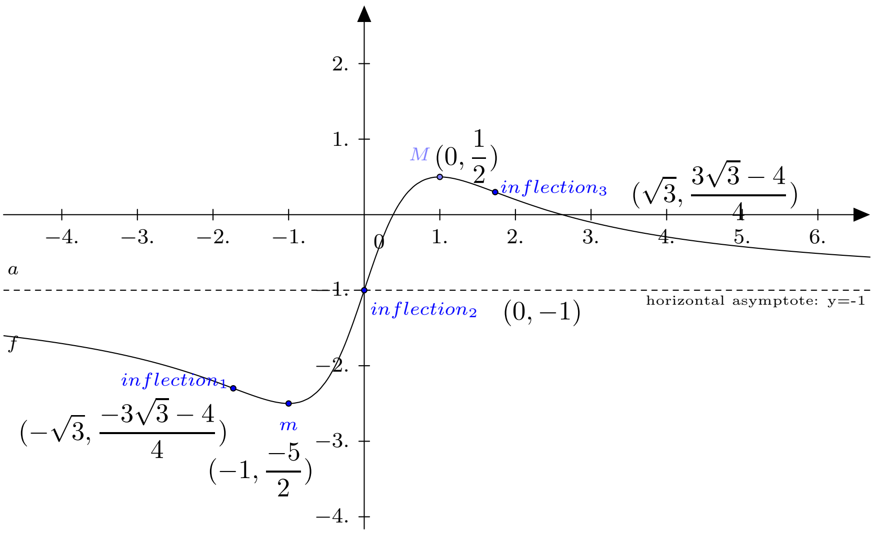
vertical asymptote:
Because f is continuous on real line, there is no vertical asymptote.

horizontal asymptote:
$\lim_{x\to\infty} \frac{-x^2+3x-1}{x^2+1} = -1$
$\lim_{x\to-\infty} \frac{-x^2+3x-1}{x^2+1} = -1$
$\Rightarrow y = -1$ is the horizontal asymptote.

其中一項的原因跟答案正確給2分，剩下的為1分。

(d) Sketch the graph of the function. Indicate, if any, where it is increasing/decreasing, where it concaves upward/downward, all relative maxima/minima, inflection points and asymptotic line(s) (if any). (4%)

Solution:



horizontal asymptote:	1 point
local maximum and local minimum:	total 1 point
兩端反曲性:	2 points

6. (12%) An elliptic billiard table (橢圓形撞球桌) is shaped by the equation

$$\frac{x^2}{4} + y^2 = 1.$$

A billiard ball is located at the $Q(1,0)$. Find all points P on the boundary of the elliptic billiard table such that the billiard ball will roll from Q to P and bounce back to Q again. (We assume that the angle of incidence is equal to the angle of bouncing back.)

Solution:

Let point P is $P(x,y) \Rightarrow$ The slope of $\overline{PQ} = \dfrac{y-0}{x-1} = \dfrac{y}{x-1}$. (1 point)

From $\dfrac{x^2}{4} + y^2 = 1$, we differentiate both sides with respect to x.

Then, we get $\dfrac{x}{2} + 2y \cdot y' = 0$.

Therefore, we find that the slope of P is $y' = \dfrac{-x}{4y}$. (3 points)

Case 1: $x \neq 1, y \neq 0$

Then, $\dfrac{y}{x-1} \cdot \dfrac{-x}{4y} = -1$. (1 point)

$\Rightarrow -x = -4x + 4 \quad \Rightarrow x = \dfrac{4}{3}$

We put $x = \dfrac{4}{3}$ back to $\dfrac{x^2}{4} + y^2 = 1 \quad \Rightarrow y = \pm\dfrac{\sqrt{5}}{3}$.

The points are $(\frac{4}{3}, \frac{\sqrt{5}}{3})$ and $(\frac{4}{3}, -\frac{\sqrt{5}}{3})$. (4 points)

Case 2. $x = 1$

From $\dfrac{x^2}{4} + y^2 = 1$, $y = \pm\dfrac{\sqrt{3}}{2}$. (1 point)

However, under $x = 1$, the slope $y' = \dfrac{-1}{4y} \neq 0 \quad \Rightarrow$ Two lines are NOT orthogonal.

Case 3. $y = 0$

From $\dfrac{x^2}{4} + y^2 = 1$, $x = \pm 2$.

$\overline{PQ} = 0$, the tangent lines of point P are vertical lines $\quad\Rightarrow\quad$ There are orthogonal to each other.

The points are $(2,0)$ and $(-2,0)$. (2 points)

Points P are $(\frac{4}{3}, \frac{\sqrt{5}}{3})$, $(\frac{4}{3}, -\frac{\sqrt{5}}{3})$, $(2,0)$ and $(-2,0)$.

7. (14%) A circular cone frustum-shaped lampcover (**正圓錐台形狀的燈罩**) is made from an annulus piece of paper by cutting out some part of it and joining the edges $\overline{AB}$ and $\overline{CD}$ as Figure 3. Find the maximum enclosed volume of such a lampcover.

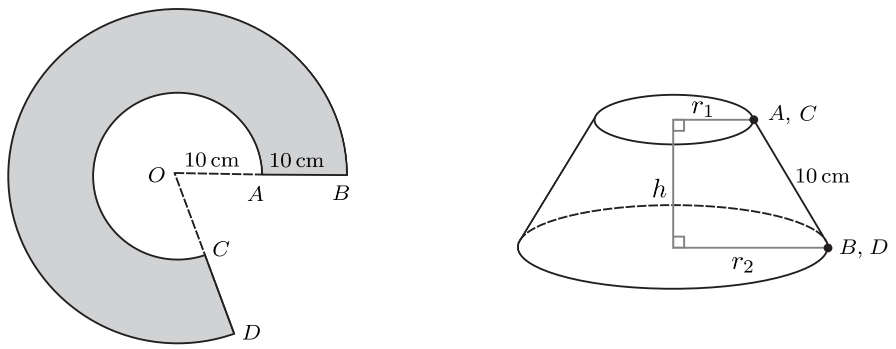


Figure 3: Make a frustum-shaped lampcover, where $\overline{OA} = 10\,\text{cm}$ and $\overline{AB} = 10\,\text{cm}$.

Remark that the volume of a circular cone frustum is $V = \dfrac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2)$, where h is the height of the frustum, and r_1, r_2 are the radii of the two bases.

Solution 1. See Figure 4. The similarity tells us that $r_2 = 2r_1$ and $r_1^2 + h^2 = 10^2$.

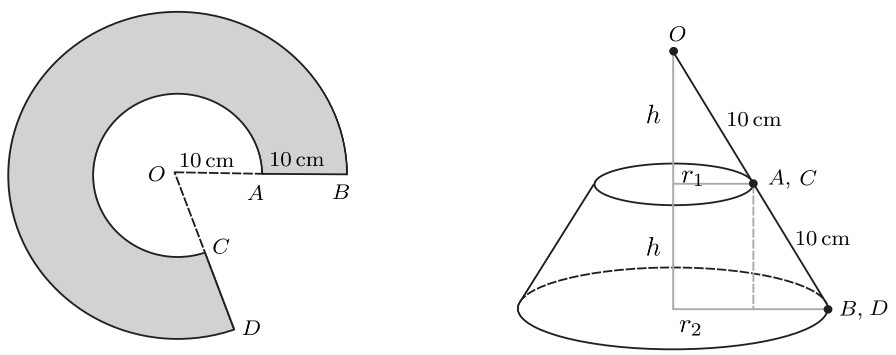


Figure 4: Find relations between r_1, r_2, and h.

So the volume of a circular cone frustum is

$$\begin{aligned} V(h) &= \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2) = \frac{\pi h}{3}(r_1^2 + r_1 \cdot 2r_1 + (2r_1)^2) \\ &= \frac{\pi h}{3} \cdot 7r_1^2 = \frac{7\pi}{3} h(10^2 - h^2) = \frac{7\pi}{3}(100h - h^3), \end{aligned}$$

where $h \in [0, 10]$. In order to find the maximum volume of $V(h)$, we compute

$$V'(h) = \frac{7\pi}{3}(100 - 3h^2) = 0 \Rightarrow h = \frac{10}{\sqrt{3}}.$$

We compare the following values:

$$V(0) = 0, \qquad V(10) = 10, \qquad V\left(\frac{10}{\sqrt{3}}\right) = \frac{7\pi}{3} \cdot \frac{10}{\sqrt{3}}\left(10^2 - \frac{10^2}{3}\right) = \frac{14000\pi}{9\sqrt{3}}.$$

Hence the maximum volume is $\dfrac{14000\pi}{9\sqrt{3}}\,\text{cm}^3$.

Solution 2. See Figure 5. The similarity tells us that $r_2 = 2r_1$ and $r_1^2 + h^2 = 10^2 \Rightarrow h = \sqrt{10^2 - r_1^2} = \sqrt{100 - r_1^2}$.

So the volume of a circular cone frustum is

$$\begin{aligned} V(r_1) &= \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2) = \frac{\pi}{3}\sqrt{100 - r_1^2}(r_1^2 + r_1(2r_1) + (2r_1)^2) \\ &= \frac{7\pi}{3} r_1^2 \sqrt{100 - r_1^2}, \end{aligned}$$

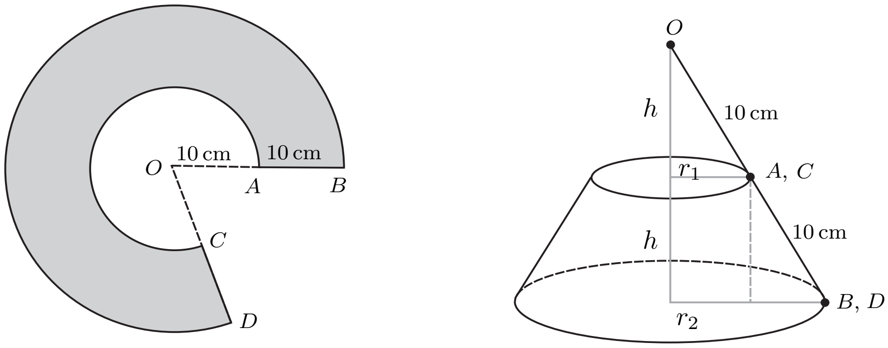


Figure 5: Find relations between r_1, r_2, and h.

where $r_1 \in [0, 10]$. In order to find the maximum volume of $V(r_1)$, we compute

$$\begin{aligned} V'(r_1) &= \frac{7\pi}{3}\left(2r_1\sqrt{100-r_1^2} + r_1^2 \cdot \frac{-2r_1}{2\sqrt{100-r_1^2}}\right) \\ &= \frac{7\pi}{3}\left(\frac{2r_1(100-r_1^2)-r_1^3}{\sqrt{100-r_1^2}}\right) = \frac{7\pi}{3}\left(\frac{r_1(-3r_1^2+200)}{\sqrt{100-r_1^2}}\right) = 0, \end{aligned}$$

then we get the critical points are $r_1 = 0$ and $r_1 = \sqrt{\frac{200}{3}} = \frac{10}{3}\sqrt{6}$. We compare the following values:

$$V(0) = 0, \qquad V(10) = 0, \qquad V\left(\frac{10}{3}\sqrt{6}\right) = \frac{7\pi}{3} \cdot \frac{200}{3}\sqrt{100 - \frac{200}{3}} = \frac{14000\pi}{9\sqrt{3}}.$$

Hence the maximum volume is $\frac{14000\pi}{9\sqrt{3}}$ cm^3.

Solution 3. See Figure 6. The similarity tells us that $r_1 = \frac{1}{2}r_2$ and $r_2^2 + (2h)^2 = 20^2 \Rightarrow h = \frac{\sqrt{20^2 - r_2^2}}{2} = \frac{\sqrt{400 - r_2^2}}{2}$.

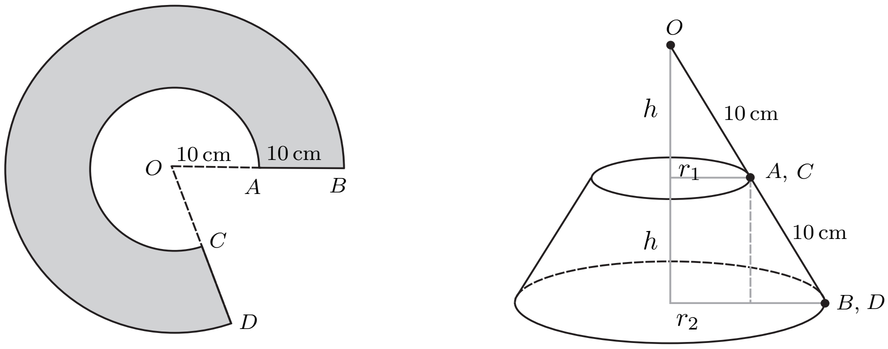


Figure 6: Find relations between r_1, r_2, and h.

So the volume of a circular cone frustum is

$$\begin{aligned} V(r_2) &= \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2) = \frac{\pi}{3} \cdot \frac{\sqrt{400-r_2^2}}{2}\left(\left(\frac{1}{2}r_2\right)^2 + \left(\frac{1}{2}r_2\right)r_2 + r_2^2\right) \\ &= \frac{\pi}{3} \cdot \frac{\sqrt{400-r_2^2}}{2} \cdot \frac{7r_2^2}{4} = \frac{7\pi}{24}r_2^2\sqrt{400-r_2^2}, \end{aligned}$$

where $r_2 \in [0, 20]$. In order to find the maximum volume of $V(r_2)$, we compute

$$\begin{aligned} V'(r_2) &= \frac{7\pi}{24}\left(2r_2\sqrt{400-r_1^2} + r_2^2 \cdot \frac{-2r_2}{2\sqrt{400-r_1^2}}\right) \\ &= \frac{7\pi}{24}\left(\frac{2r_2(400-r_2^2)-r_2^3}{\sqrt{400-r_2^2}}\right) = \frac{7\pi}{24}\left(\frac{r_2(-3r_2^2+800)}{\sqrt{400-r_2^2}}\right) = 0, \end{aligned}$$

then we get the critical points are $r_2 = 0$ and $r_2 = \sqrt{\dfrac{800}{3}} = \dfrac{20}{3}\sqrt{6}$. We compare the following values:

$$V(0) = 0, \qquad V(20) = 0, \qquad V\left(\frac{20}{3}\sqrt{6}\right) = \frac{7\pi}{24} \cdot \frac{800}{3}\sqrt{400 - \frac{800}{3}} = \frac{14000\pi}{9\sqrt{3}}.$$

Hence the maximum volume is $\dfrac{14000\pi}{9\sqrt{3}}$ cm^3.

Solution 4. See Figure 7. Let θ be the angle of the annulus papers, then we know that

$$\begin{aligned}
&2r_1\pi = 10\theta \Rightarrow r_1 = \frac{5\theta}{\pi},\\
&2r_2\pi = 20\theta \Rightarrow r_2 = \frac{10\theta}{\pi},\\
&h^2 + r_1^2 = 10^2 \Rightarrow h = \sqrt{10^2 - r_1^2} = \sqrt{100 - \frac{25\theta^1}{\pi^2}} = \frac{5}{\pi}\sqrt{4\pi - \theta^2}.
\end{aligned}$$

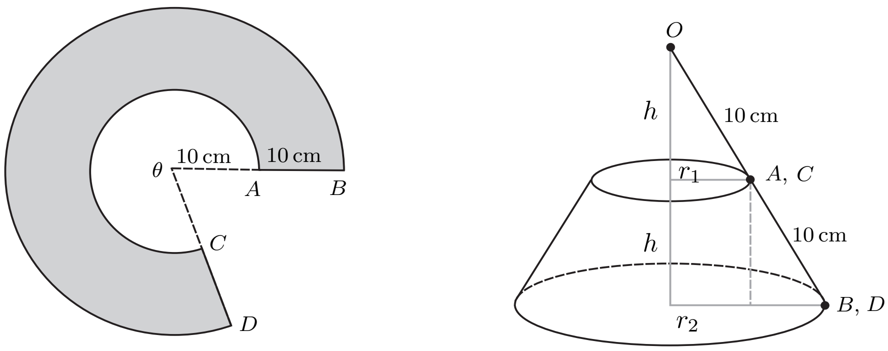


Figure 7: Find relations between θ, r_1, r_2, and h.

So the volume of a circular cone frustum is

$$\begin{aligned}
V(\theta) &= \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2) = \frac{\pi}{3}\cdot\frac{5}{\pi}\sqrt{4\pi^2 - \theta^2}\cdot 7r_1^2\\
&= \frac{\pi}{3}\cdot\frac{5}{\pi}\sqrt{4\pi^2 - \theta^2}\cdot 7\left(\frac{5\theta}{\pi}\right)^2 = \frac{875\theta^2}{3\pi^2}\sqrt{4\pi - \theta^2},
\end{aligned}$$

where $\theta \in [0, 2\pi]$. In order to find the maximum volume of $V(\theta)$, we compute

$$\begin{aligned}
V'(\theta) &= \frac{875}{3\pi^2}\left(2\theta\sqrt{4\pi^2 - \theta^2} + \theta^2\cdot\frac{-2\theta}{2\sqrt{4\pi^2 - \theta^2}}\right) = \frac{875}{3\pi^2}\left(\frac{2\theta(4\pi^2 - \theta^2) - \theta^3}{\sqrt{4\pi^2 - \theta^2}}\right)\\
&= \frac{875}{3\pi^2}\left(\frac{-\theta(3\theta^2 - 8\pi^2)}{\sqrt{4\pi^2 - \theta^2}}\right) = 0,
\end{aligned}$$

then we get the critical points are $\theta = 0$, and $\theta = \sqrt{\dfrac{8}{3}\pi^2} = \dfrac{2}{3}\sqrt{6}\pi$. We compare the following values:

$$\begin{aligned}
&V(0) = 0, \qquad V(20) = 0,\\
&V\left(\frac{2}{3}\sqrt{6}\pi\right) = \frac{875}{3\pi^2}\cdot\frac{8}{3}\pi^2\sqrt{4\pi - \frac{8}{3}\pi^2} = \frac{875\cdot 8}{9}\cdot\frac{2\pi}{\sqrt{3}} = \frac{14000\pi}{9\sqrt{3}}.
\end{aligned}$$

Hence the maximum volume is $\dfrac{14000\pi}{9\sqrt{3}}$ cm^3.

Solution 5. See Figure 8. Let ϕ be the cutting angle of the annulus paper, then we know that

$$\begin{aligned}
&2r_1\pi = 10(2\pi - \phi) \Rightarrow r_1 = \frac{5(2\pi - \theta)}{\pi} = 5\left(2 - \frac{\phi}{\pi}\right) = 10 - \frac{5\phi}{\pi},\\
&2r_2\pi = 20(2\pi - \phi) \Rightarrow r_2 = \frac{10(2\pi - \theta)}{\pi} = 10\left(2 - \frac{\phi}{\pi}\right) = 20 - \frac{10\phi}{\pi},\\
&h^2 + r_1^2 = 10^2 \Rightarrow h = \sqrt{10^2 - r_1^2} = \sqrt{100 - \left(10 - \frac{5\phi}{\pi}\right)^2} = \frac{5}{\pi}\sqrt{4\pi\phi - \phi^2}.
\end{aligned}$$

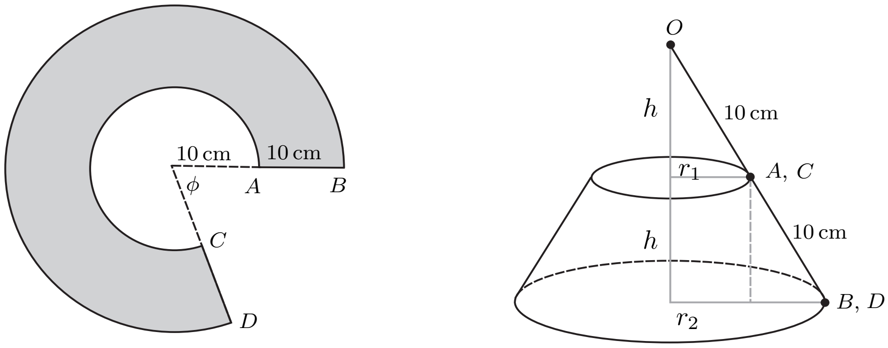


Figure 8: Find relations between ϕ, r_1, r_2, and h.

So the volume of a circular cone frustum is

$$
\begin{aligned}
V(\phi) &= \frac{\pi h}{3}(r_1^2 + r_1 r_2 + r_2^2) = \frac{\pi}{3} \cdot \frac{5}{\pi}\sqrt{4\pi\phi - \phi^2} \cdot 7\left(\frac{5}{\pi}(2\pi - \phi)\right)^2 \\
&= \frac{875}{3\pi^2}\sqrt{\phi(4\phi - \phi)}(2\pi - \phi)^2 = \frac{875}{3\pi^2}\sqrt{4\pi\phi - \phi^2}(2\pi - \phi)^2,
\end{aligned}
$$

where $\theta \in [0, 2\pi]$. In order to find the maximum volume of $V(\theta)$, we compute

$$
\begin{aligned}
V'(\theta) &= \frac{875}{3\pi^2}\left(\frac{1}{2}\frac{4\pi - 2\phi}{\sqrt{4\pi\phi - \phi^2}}(2\pi - \phi)^2 + \sqrt{4\pi\phi - \phi^2}(-4\pi + 2\phi)\right) \\
&= \frac{875}{3\pi^2}\frac{(2\pi - \phi)}{\sqrt{4\pi\phi - \phi^2}}\left((2\pi - \phi)^2 - 2(4\pi\phi - \phi^2)\right) \\
&= \frac{875}{3\pi^2}\frac{(2\pi - \phi)}{\sqrt{4\pi\phi - \phi 2}}\left(4\pi^2 - 12\pi\phi + 3\phi^2\right).
\end{aligned}
$$

then we get the critical point is $\phi = \dfrac{12\pi - \sqrt{144\pi^2 - 48\pi^2}}{6} = 2\pi - \dfrac{2}{3}\sqrt{6}$. We compare the following values:

$$
\begin{aligned}
&V(0) = 0, \qquad V(20) = 0, \\
&V\left(2\pi - \frac{2}{3}\sqrt{6}\right) = \frac{875}{3\pi^2}\sqrt{\left(2\pi - \frac{2}{3}\sqrt{6}\pi\right)\left(2\pi + \frac{2}{3}\sqrt{6}\pi\right)}\left(\frac{8}{3}\pi\right)^2 = \frac{14000\pi}{9\sqrt{3}}.
\end{aligned}
$$

Hence the maximum volume is $\dfrac{14000\pi}{9\sqrt{3}}$ cm^3.

評分標準

Let $\theta = \angle AOC$ be the angle of the major sector and $\phi = 2\pi - \theta$.
Let α be the angle between $\overline{AB}$ and the base of the lampcover.

a. [4pt] Relationship between r_1, r_2 and h
   (a) [2pt] $r_2 = 2r_1$ or $r_1 = \dfrac{5\theta}{\pi}, r_2 = \dfrac{10\theta}{\pi}$ or $r_1 = \dfrac{5}{\pi}(2\pi - \phi), r_2 = \dfrac{10}{\pi}(2\pi - \phi)$ or $r_1 = 10\cos\alpha, r_2 = 20\cos\alpha$
   (b) [2pt] $h^2 + (r_2 - r_1)^2 = 10^2$ or $h = \dfrac{5}{\pi}\sqrt{4\pi^2 - \theta^2}$ or $h = \dfrac{5}{\pi}\sqrt{4\pi\phi - \phi^2}$ or $h = 10\sin\alpha$

b. [2pt] Form the target function V
   (a) $V = \dfrac{7\pi}{3}\sqrt{100 - r_1^2} \cdot r_1^2$
   (b) $V = \dfrac{7\pi}{24}\sqrt{400 - r_2^2} \cdot r_2^2$
   (c) $V = \dfrac{7\pi}{3}h(100 - h^2)$
   (d) $V = \dfrac{875\theta^2}{3\pi}\sqrt{4 - \dfrac{\theta^2}{\pi^2}}$
   (e) $V = \dfrac{875}{3\pi^2}\sqrt{4\pi\phi - \phi^2} \cdot (2\pi - \phi)^2$

(f) $V = \frac{7000\pi}{3}(\sin\alpha - \sin^3\alpha)$

c. [3pt] Take derivative

(a) $\frac{dV}{dr_1} = \frac{7\pi}{3}\left(\frac{-r_1^3}{\sqrt{100-r_1^2}} + 2r_1\sqrt{100-r_1^2}\right)$

(b) $\frac{dV}{dr_2} = \frac{7\pi}{24}\left(\frac{-r_2^3}{\sqrt{400-r_2^2}} + 2r_2\sqrt{400-r_2^2}\right)$

(c) $\frac{dV}{dh} = \frac{7\pi}{3}(100-3h^2)$

(d) $\frac{dV}{d\theta} = \frac{875\theta}{3\pi^2} \cdot \frac{8\pi^2 - 3\theta^2}{\sqrt{4\pi^2-\theta^2}}$

(e) $\frac{dV}{d\phi} = \frac{875}{3\pi^2} \cdot \frac{(2\pi-\phi)(3\phi^2 - 12\pi\phi + 4\pi^2)}{\sqrt{2\pi\phi - \phi^2}}$

(f) $\frac{dV}{d\alpha} = \frac{7000\pi}{3}(\cos\alpha - 3\sin^2\alpha\cos\alpha)$

d. [2pt] Solve $V' = 0$

(a) $r_1 = \frac{10}{3}\sqrt{6}$

(b) $r_2 = \frac{20}{3}\sqrt{6}$

(c) $h = \frac{10}{3}\sqrt{3}$

(d) $\theta = \sqrt{\frac{8}{3}}\pi$

(e) $\phi = 2\pi(1 - \sqrt{\frac{2}{3}})$

(f) $\alpha = \sin^{-1}\sqrt{\frac{1}{3}}$

e. [2pt] Check: volume=max

f. [1pt] Calculate: $V_{max} = \frac{14000\pi}{9\sqrt{3}}$

Note: If $r_1 = 10$ is assumed (which is wrong, of course), one can get 7points at most.

8. (12%) An object at rest with mass m is dragged along a horizontal plane by a force acting along a rope attached to the object so that the object remains at rest as Figure 9.

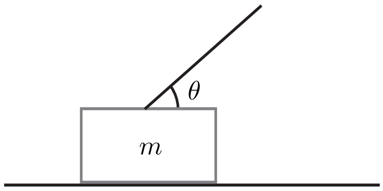


Figure 9: Drag an object with $0 \le \theta \le \frac{\pi}{2}$.

If the rope makes an angle θ with a plane, where $0 \le \theta \le \frac{\pi}{2}$, then the magnitude of the force F will satisfy the equation

$$\mu(mg - F\sin\theta) = F\cos\theta,$$

where μ is a positive constant called the coefficient of static friction and g is the gravitational constant. For what value of θ is F smallest?

Solution:

Solution 1. Since $F = F(\theta)$, we implicit differentiation the equation with respect to θ and get

$$\mu(-F'(\theta)\sin\theta - F(\theta)\cos\theta) = F'(\theta)\cos\theta - F(\theta)\sin\theta. \quad (\textbf{4 points}) \tag{1}$$

We will solve θ such that $F'(\theta) = 0$, which means

$$\mu(-F(\theta)\cos\theta) = -F(\theta)\sin\theta \Rightarrow F(\theta)(-\mu\cos\theta + \sin\theta) = 0.$$

Since $F(\theta) \neq 0$, we have $\mu\cos\theta = \sin\theta \Rightarrow \tan\theta = \mu \Rightarrow \theta = \tan^{-1}\mu$. **(5 points)**

From (1), we get

$$F'(\theta)(\cos\theta + \mu\sin\theta) = F(\theta)(\sin\theta - \mu\cos\theta) \Rightarrow F'(\theta) = \frac{F(\theta)(\sin\theta - \mu\cos\theta)}{\cos\theta + \mu\sin\theta}. \tag{2}$$

- **If $\theta < \tan^{-1}\mu$, then** $\tan\theta = \frac{\sin\theta}{\cos\theta} < \mu \Rightarrow \sin\theta - \mu\cos\theta < 0$. **From (2), we know that** $F'(\theta) < 0$.
- **If $\theta > \tan^{-1}\mu$, then** $\tan\theta = \frac{\sin\theta}{\cos\theta} > \mu \Rightarrow \sin\theta - \mu\cos\theta > 0$. **From (2), we know that** $F'(\theta) > 0$.

Hence $\theta = \tan^{-1}\mu$ will attain the local (and hence global) minimum value of $F(\theta)$. (3 points)

Solution 2. Since $\mu(mg - F\sin\theta) = F\cos\theta \Rightarrow F(\mu\sin\theta + \cos\theta) = \mu mg$, we know that

$$F(\theta) = \frac{\mu mg}{\mu\sin\theta + \cos\theta}. \quad (\textbf{4 points})$$

We compute

$$F'(\theta) = -\frac{\mu mg}{(\mu\sin\theta + \cos\theta)^2} \cdot (\mu\cos\theta - \sin\theta).$$

We solve $F'(\theta) = 0$ and get $\tan\theta = \mu \Rightarrow \theta = \tan^{-1}\mu$. **(5 points)**

- **If $\theta < \tan^{-1}\mu$, then** $\tan\theta = \frac{\sin\theta}{\cos\theta} < \mu \Rightarrow \mu\cos\theta - \sin\theta > 0$. **We know that** $F'(\theta) < 0$.
- **If $\theta > \tan^{-1}\mu$, then** $\tan\theta = \frac{\sin\theta}{\cos\theta} > \mu \Rightarrow \mu\cos\theta - \sin\theta < 0$. **We know that** $F'(\theta) > 0$.

Hence $\theta = \tan^{-1}\mu$ will attain the local (and hence global) minimum value of $F(\theta)$. (3 points)

注意：如果沒有說明為何$\theta = \tan^{-1}\mu$ 為極小值扣兩分