1041微甲06-10班期末考解答和評分標準

1. (14%)

   (a) (7%) Find the indefinite integrals $\int \frac{1}{\sqrt{x^2+1}}dx$ (5%) and $\int \frac{1}{x+2}dx$ (2%). (You may use the integral formula of $\int \sec\theta\ d\theta$.)

   (b) (4%) Find the value of the constant a for which the improper integral

   $$\int_0^\infty \left(\frac{1}{\sqrt{x^2+1}} - \frac{a}{x+2}\right) dx \quad \text{converges.}$$

   (c) (3%) Evaluate the improper integral for this a.

Solution:

(a)(7pts)

[part1–(5pts)]

$$\int \frac{1}{\sqrt{x^2+1}}\ dx$$

let $x = \tan\theta\ \Rightarrow dx{=}\sec^2\theta\ d\theta$ (2pts)

$$\int \frac{1}{\sqrt{x^2+1}}\ dx$$

$$= \int \frac{\sec^2\theta}{\sec\theta}\ d\theta$$

$$= \int \sec\theta\ d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln|\sqrt{x^2+1} + x| + c$$

[part2–(2pts)]

$$\int \frac{1}{x+2}\ dx$$

$$= \ln|x+2| + c$$

(b)(4pts)

$$\int_0^\infty \left(\frac{1}{\sqrt{x^2+1}} - \frac{a}{x+2}\right)dx$$

$$= \lim_{t\to\infty} \int_0^t \left(\frac{1}{\sqrt{x^2+1}} - \frac{a}{x+2}\right)dx$$

consider the indefinite integral

$$\int \left(\frac{1}{\sqrt{x^2+1}} - \frac{a}{x+2}\right)dx$$

$$= \ln|\sqrt{x^2+1} + x| - a\ln|x+2| + c$$

Let I=$\int_0^\infty (\frac{1}{\sqrt{x^2+1}} - \frac{a}{x+2})dx$

I=$\int_0^\infty (\frac{1}{\sqrt{x^2+1}} - \frac{a}{x+2})dx$

$=\lim_{t\to\infty} \int_0^t (\frac{1}{\sqrt{x^2+1}} - \frac{a}{x+2})dx$

$=\lim_{t\to\infty} (\ln(\sqrt{t^2+1}+t) - \ln 1 - a\ln(t+2) + a\ln 2)$

$=\lim_{t\to\infty} \ln(\frac{\sqrt{t^2+1}+t}{(t+2)^a}) + a\ln 2$ ———————————————— (2pts)

Thus we only need to consider $\lim_{t\to\infty} \ln\frac{\sqrt{t^2+1}+t}{(t+2)^a}$

let L= $\lim_{t\to\infty} \ln\frac{\sqrt{t^2+1}+t}{(t+2)^a}$

I=$\int_0^\infty (\frac{1}{\sqrt{x^2+1}} - \frac{a}{x+2})dx$=ln L+ $\ln 2$

(case1) $a > 1$

L=0

I= ln L $+a\ln 2 \to -\infty$ dverges

(case2) $a < 1$ then L$\to \infty$, and I= ln L $+a\ln 2 \to \infty$ dverges

(case1)and(case2)———————————————————— (1pts)

(case3) $a = 1$

L= $\lim_{t\to\infty} \ln\frac{\sqrt{t^2+1}+t}{(t+2)} = 2$

I=ln L+ $\ln 2 = \ln 2 + \ln 2 = 2\ln 2$

Thus I converges when a=1———————————————— (1pts)

(c)(3pts)

When a=1

L= $\lim_{t\to\infty} \ln\frac{\sqrt{t^2+1}+t}{(t+2)} = 2$———————————————— (2pts)

I=ln L+ $\ln 2 = \ln 2 + \ln 2 = 2\ln 2$————————————— (1pts)

2. (12%) Find the following indefinite integrals:

   (a) (6%) $\displaystyle\int \frac{3t^2+t+4}{t^3+t}\,dt.$

   (b) (6%) $\displaystyle\int \cos\sqrt{x}\,dx$

Solution:

(a) $\dfrac{3t^2+t+4}{t^3+t} = \dfrac{A}{t} + \dfrac{Bt+C}{t^2+1}$(1 point)

$\Rightarrow A = 4, B = -1, C = 1$(1 point)

$\displaystyle\int \frac{4}{t} + \frac{-t+1}{t^2+1} dt = 4\ln|t| - \frac{1}{2}\ln(t^2+1) + \tan^{-1} t + c$(1 pointt for each)

(b) Let $\sqrt{x} = u, \Rightarrow dx = 2udu.$

$$\begin{aligned}\int \cos\sqrt{x}dx &= 2\int u\cos udu \text{(2 points)}\\ &= 2\int ud\sin\\ &= 2(u\sin u - \int \sin udu)\text{(2 points)}\\ &= 2(u\sin u + \cos u) + c\\ &= (2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x}) + c\text{(1 point for each)}\end{aligned}$$

3. (12%)

(a) (10%) Solve the initial-value problem: $\frac{dx}{dt} = (a-x)(b-x)$, where $a > b > 0$, for $x = x(t)$ with $x(0) = 0$.

(b) (2%) Find $\lim_{t\to\infty} x(t)$.

Solution:

(a) $$\frac{dx}{(a-x)(b-x)} = dt$$
$$\Rightarrow \int \frac{dx}{(a-x)(b-x)} = \int 1dt$$
$$\Rightarrow \frac{1}{b-a}\int (\frac{1}{a-x} - \frac{1}{b-x})dx = t + C$$
$$\Rightarrow \frac{1}{b-a}(-\ln|a-x| + \ln|b-x|) = t + C$$
$$\Rightarrow \frac{1}{a-b}\ln\left|\frac{a-x}{b-x}\right| = t + C$$
$$\because x(0) = 0$$
$$\therefore C = \frac{1}{a-b}\ln\left|\frac{a}{b}\right| = \frac{1}{a-b}\ln\frac{a}{b}$$
$$\Rightarrow \ln\left|\frac{a-x}{b-x}\right| = (a-b)t + \ln\frac{a}{b}$$
$$\Rightarrow \frac{a-x}{b-x} = \frac{a}{b}e^{(a-b)t}$$
$$\Rightarrow abe^{(a-b)t} - ae^{(a-b)t} = ab - bx$$
$$\Rightarrow x(t) = \frac{ab(e^{(a-b)t} - 1)}{ae^{(a-b)t} - b}$$

(b) $$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} \frac{ab(e^{(a-b)t} - 1)}{ae^{(a-b)t} - b} = \lim_{t\to\infty} \frac{ab(a-b)e^{(a-b)t}}{a(a-b)e^{(a-b)t}} = b$$

評分標準

(a) **變數分離得** 2 **分**

積分出來得 4 **分**

把 C **求出來得** 2 **分**

帶回 C **並解出** x **得** 2 **分**

(b) **沒過程不給分**

4. (12%)

(a) (10%) Solve the initial-value problem: $xy' - y = x^2 \sin x$, with $y(\pi) = 0$.

(b) (2%) Find $\lim_{x\to 0^+} \frac{y(x)}{x^2}$.

Solution:

(a) The linear differential equation is $y' - \frac{1}{x}y = x\sin x$. We multiply the integrating factor

$$e^{\int -\frac{1}{x}\,dx} = e^{-\ln|x|} = x^{-1} \quad \text{(積分因子有算出來得 5 分，有錯整題最多給 2 分。)}$$

on both sides of the differential equation and get

$$\frac{1}{x}y' - \frac{1}{x^2}y = \sin x \Rightarrow \frac{d}{dx}\left(\frac{1}{x}y\right) = \sin x \Rightarrow \frac{1}{x}y = \int \sin x\,dx = -\cos x + C.$$

So $y(x) = x(-\cos x + C)$. (寫到這裡可得 8 分。) The initial condition $y(\pi) = 0$ implies $0 = \pi(1 + C)$ and we get $C = -1$, (解對積分常數再得 1 分。) so the solution of the differential equation is

$$y(x) = x(-\cos x - 1). \quad \text{(最後完整的函數寫出再得 1 分。)}$$

(b) The limit is

$$\lim_{x\to 0^+} \frac{y(x)}{x^2} = \lim_{x\to 0^+} \frac{x(-\cos x - 1)}{x^2} = \lim_{x\to 0^+} \frac{-\cos x - 1}{x} = -\infty.$$

- 注意到，若用以上觀察，寫「$-\infty$」或「不存在」或「發散」都可以算對，得 2 分。 若是用羅必達法則計算該極限，最後的答案只能寫「$-\infty$」，而不能寫「不存在」或「發散」。 因為使用羅必達法則後，若分子、分母各自微分後的極限「不存在」或「發散」，並不能對原極限下結論； 於是判定你對羅必達法則使用錯誤或是沒有正確理解，無法給分。 課本第 302 頁指出，使用羅必達後，唯有「極限存在」或是「$\pm\infty$」這種狀況可以反推原極限的結果。

5. (10%) Evaluate $\lim_{x\to 0} \frac{1}{x} \int_0^x (1-\tan t)^{1/t}\,dt$.

Solution:

First, note that $\lim_{x\to 0} \frac{\int_0^x (1-\tan t)^{\frac{1}{t}}\,dt}{x}$ is of the form $\frac{0}{0}$.
We can use l'Hospital's rule to evaluate the limit.

$\lim_{x\to 0} \frac{\int_0^x (1-\tan t)^{\frac{1}{t}}\,dt}{x} = \lim_{x\to 0}(1-\tan x)^{\frac{1}{x}}$ By fundamental theorem of calculus. (5 pts)

Then we take log to evaluate the last limit.

$\lim_{x\to 0} \frac{\ln(1-\tan x)}{x}$ is of the form $\frac{0}{0}$
$= \lim_{x\to 0} \frac{-\sec^2 x}{1-\tan x} = -1$ by l'Hospital's rule

Thus we have $\lim_{x\to 0} \frac{\int_0^x (1-\tan t)^{\frac{1}{t}}\,dt}{x} = e^{-1} = \frac{1}{e}$ (5 pts)

Note $\lim_{x\to 0} \frac{\int_0^x (1-\tan t)^{\frac{1}{t}}\,dt}{x} \neq f'(0)$

where $f(x) = \int_0^x (1-\tan t)^{\frac{1}{t}}\,dt$

FTC:If $f(x) = \int_0^x g(t)\,dt$ for $x \in [0,a]$
then $f'(x) = g(x)$ for $x \in (0,a)$ but **not** for $x = 0, a$

6. (12%)

(a) (4%) Show that the area of an ellipse with the semi-major axis of length a and the semi-minor axis of length b is $ab\pi$. See Figure 1(a).

(b) (8%) A toothpaste tube is modeled in Figure 1(b).

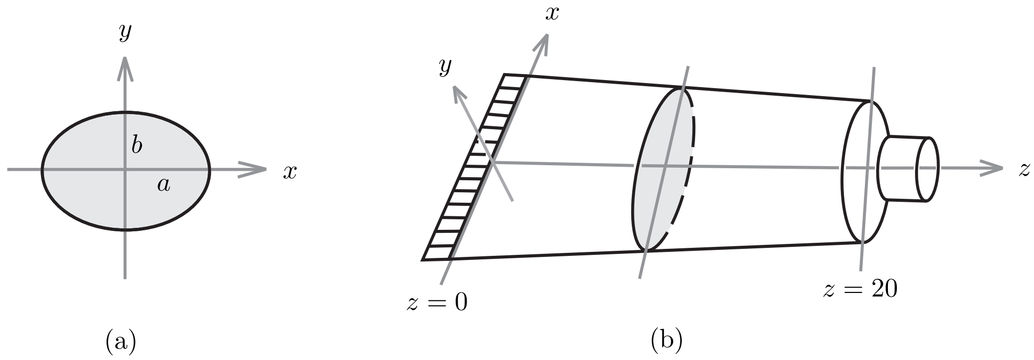

Figure 1: (a) The area of an ellipse is $ab\pi$. (b) Find the volume of the modeled toothpaste tube.

- One side is flat and is located at $-\pi \leq x \leq \pi, y = 0, z = 0$.
- The other side is a circle with radius 2, so the equation of the circle is $x^2 + y^2 = 4, z = 20$.
- Each cross-section for $0 < z < 20$ is an ellipse with the semi-major axis of length a and the semi-minor axis of length b, where

$$a = \pi + (2 - \pi)\frac{z}{20} \quad \text{and} \quad b = \frac{1}{10}z.$$

Find the volume of the modeled toothpaste tube with $0 \leq z \leq 20$.

Solution:

(a)

Ellipse equation: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ =¿ $y = \pm b\sqrt{1 - \frac{x^2}{a^2}}$ (1%)

Aera

$= 4\int_0^a b\sqrt{1 - \frac{x^2}{a^2}}dx$ (1%)$= \frac{4b}{a}\int_0^a \sqrt{a^2 - x^2}dx$

$= 4ab\int_0^{\frac{\pi}{2}} \cos^2\theta d\theta$ (let $x = a\sin\theta,\ dx = a\cos\theta d\theta$) (1%)

$= 2ab\int_0^{\frac{\pi}{2}} (1 + \cos 2\theta)d\theta$ (1%)

$= 2ab(\theta + \frac{1}{2}\sin 2\theta)|_0^{\frac{\pi}{2}}$

$= ab\pi$ (1%)

(b)

Volume

$= \int_0^{20} \pi(\pi + (2 - \pi)\frac{z}{20})\frac{1}{10}zdz$ (4%)

$= \int_0^{20} \frac{1}{10}\pi^2 z + \pi(2 - \pi)\frac{z^2}{200}dz$

$= (\frac{\pi^2}{20}z^2 + \frac{\pi(2 - \pi)}{600}z^3)|_0^{20}$ (2%)

$= \frac{20}{3}\pi^2 + \frac{80}{3}\pi$ (2%)

7. (12%)

(a) (3%) Find all intersection points of the two curves $r = \sqrt{2}\sin\theta$ and $r^2 = \cos 2\theta$ in their polar equations.

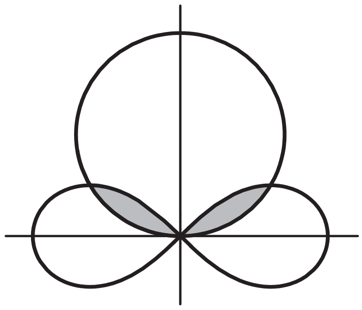

Figure 2: (a) Find all intersection points of the two curves. (b)Find the area of the shaded region.

(b) (9%) Find the area of the shaded region in Figure 2.

Solution:

(**a**). If we solve the equations $r = \sqrt{2}\sin\theta$ and $r^2 = \cos 2\theta$, we get

$$\begin{cases} r = \sqrt{2}\sin\theta \\ r^2 = \cos 2\theta \end{cases}$$

$$\Rightarrow \quad 2\sin^2\theta = \cos 2\theta = 1 - 2\sin^2\theta$$

$$\Rightarrow \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$$

$$\Rightarrow \quad r = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$$

We have found two points of intersection:

$$(r,\theta) = (\frac{1}{\sqrt{2}}, \frac{\pi}{6}) \quad \textbf{(1pt)}$$

$$(r,\theta) = (\frac{1}{\sqrt{2}}, \frac{5\pi}{6}) \quad \textbf{(1pt)}$$

However, if we plug $\theta = 0$ into $r = \sqrt{2}\sin\theta$ and $\theta = \pi/4$ into $r^2 = \cos 2\theta$,we can find one more point of intersection:

$$r = 0 \quad \textbf{(1pt)}$$

(**b**).

$$\begin{aligned} A &= 2\left(\frac{1}{2}\int_0^{\frac{\pi}{6}} (\sqrt{2}\sin\theta)^2 d\theta + \frac{1}{2}\int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2\theta d\theta\right) \quad \textbf{(6pts)} \\ &= \int_0^{\frac{\pi}{6}} (1-\cos 2\theta)d\theta + \int_{\frac{\pi}{6}}^{\frac{\pi}{4}} \cos 2\theta d\theta \\ &= \left(\theta - \frac{1}{2}\sin 2\theta\right)\Bigg|_0^{\frac{\pi}{6}} + \frac{1}{2}\sin 2\theta\Bigg|_{\frac{\pi}{6}}^{\frac{\pi}{4}} \quad \textbf{(2pts)} \\ &= \frac{\pi}{6} + \frac{1}{2} - \frac{\sqrt{3}}{2} \quad \textbf{(1pt)} \end{aligned}$$

8. (16%) The front door of a school bus is designed as in Figure 3. It is a folding door with $\overline{AB} = \overline{BO} = \dfrac{1}{2}$. The door is opened or closed by rotating $\overline{OB}$ about z-axis, while A is moving along y-axis.

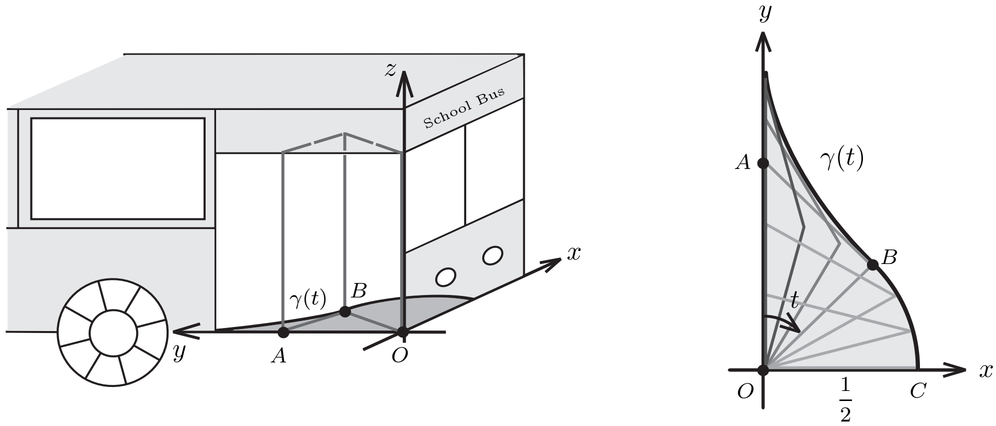

Figure 3: (a) Find the enclosed area by the curve $\gamma(t)$ and two axes. (b) Find the length of the curve $\gamma(t)$.

The region swept out by the bus door on the xy-plane is enclosed by the two axes and the curve $\gamma(t)$ parametrized by

$$\gamma(t) = (x(t), y(t)) = \begin{cases} (\sin^3 t, \cos^3 t) & \text{if } 0 \le t \le \dfrac{\pi}{4} \\ \left(\dfrac{1}{2}\sin t, \dfrac{1}{2}\cos t\right) & \text{if } \dfrac{\pi}{4} \le t \le \dfrac{\pi}{2}, \end{cases}$$

where $0 \le t \le \dfrac{\pi}{2}$ denotes the angle between the positive y-axis and $\overline{OB}$.

(a) (10%) Find the area of this region.

(b) (6%) Find the length of the curve $\gamma(t)$.

Solution:

The curve consists of two parts: a half of a branch of an astroid (for $0 \le t \le \dfrac{\pi}{4}$), and one-eighth of a circle (for $\dfrac{\pi}{4} \le t \le \dfrac{\pi}{2}$).

(a)

(1) Formulation:

$$A = \int y\,dx = \int_0^{\frac{\pi}{4}} y(t)x'(t)dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} y(t)x'(t)dt = \int_0^{\frac{\pi}{4}} \cos^3 t \cdot 3\sin^2 t \cos t dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}\cos t \cdot \frac{1}{2}\cos t dt$$
$$= \int_0^{\frac{\pi}{4}} 3\cos^4 t \sin^2 t dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}\cos^2 t dt$$

Or, noting that as y goes from 0 to 1, t varies from $\dfrac{\pi}{2}$ to 0,

$$A = \int x\,dy = \int_{\frac{\pi}{4}}^{0} x(t)y'(t)dt + \int_{\frac{\pi}{2}}^{\frac{\pi}{4}} x(t)y'(t)dt$$
$$= \int_0^{\frac{\pi}{4}} 3\cos^2 t \sin^4 t dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}\sin^2 t dt$$

Alternatively, if polar coordinates are to be used (letting the polar axis be in the $+y$ direction and θ be in the clockwise direction), $\theta = t$ for the circle part, while $\theta = \tan^{-1}(\dfrac{\sin^3 t}{\cos^3 t}) = \tan^{-1}(\tan^3 t)$, such that

$$A = \int \frac{1}{2}r^2 d\theta = \int_0^{\frac{\pi}{4}} \frac{1}{2}r^2(t)\frac{d\theta}{dt}dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}r^2(t)dt$$
$$= \int_0^{\frac{\pi}{4}} \frac{1}{2}(\sin^6 t + \cos^6 t)\frac{3\tan^2 t \sec^2 t}{1+\tan^6 t}dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}dt$$
$$= \int_0^{\frac{\pi}{4}} \frac{3}{2}\sin^2 t \cos^2 t dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}dt$$

(2) Integration and evaluation:

$$\begin{aligned}\int \cos^4 t \sin^2 t dt &= \int (\frac{1+\cos 2t}{2})(\frac{1}{4}\sin^2 2t)dt \\ &= \frac{1}{8}\int (\frac{1-\cos 4t}{2} + \sin^2 2t \cos 2t)dt \\ &= \frac{1}{16}(t - \frac{1}{4}\sin 4t + \frac{1}{3}\sin^3 t)\end{aligned}$$

$$\int \cos^2 t dt = \int (\frac{1+\cos 2t}{2})dt = \frac{1}{2}t + \frac{1}{4}\sin 2t$$

$$\begin{aligned}A = \int y dx &= \int_0^{\frac{\pi}{4}} 3\cos^4 t \sin^2 t dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{4}\cos^2 t dt \\ &= 3\cdot\frac{1}{16}(\frac{\pi}{4} - 0 + \frac{1}{3}) + \frac{1}{4}(\frac{\pi}{8} - \frac{1}{4}) = (\frac{3}{64}\pi + \frac{1}{16}) + (\frac{1}{32}\pi - \frac{1}{16}) = \frac{5}{64}\pi\end{aligned}$$

(For your reference, the other methods yields $\int y dx = (\frac{3}{64}\pi - \frac{1}{16}) + (\frac{1}{32}\pi + \frac{1}{16}) = \frac{5}{64}\pi$ and $\int \frac{1}{2}r^2 d\theta = \frac{3}{64}\pi + \frac{1}{32}\pi = \frac{5}{64}\pi$ respectively.)

• Grading policy: for the astroid part: 3% for formulation, 3% for integration and 2% for evaluation; 2% in total for the one-eighth circle part.

(b)
(1) For the astroid part:

$$L_1 = \int ds = \int_0^{\frac{\pi}{4}} \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}dt = \int_0^{\frac{\pi}{4}} \sqrt{(3\sin^2 t\cos t)^2 + (-3\cos^2 t \sin t)^2}dt \qquad (3\%)$$

$$= \int_0^{\frac{\pi}{4}} 3|\sin t\cos t|\sqrt{\sin^2 t + \cos^2 t}dt = 3\ [\frac{1}{2}\sin^2 t]_0^{\frac{\pi}{4}} = \frac{4}{3} \qquad (2\%)$$

(2) For the circle part:

$$L_2 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{(\frac{1}{2}\cos t)^2 + (-\frac{1}{2}\sin t)^2}dt = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{1}{2}dt = \frac{\pi}{8} \qquad (1\%)$$

Or directly $L_2 = \frac{1}{8}(2\cdot\pi\cdot\frac{1}{2}) = \frac{\pi}{8}$ since it is one-eighth of a circle.

Thus the total length is $L = L_1 + L_2 = \frac{3}{4} + \frac{\pi}{8}$.