1041微甲01-04班期中考解答和評分標準

1. (32%) Evaluate the following limits.

(a) (8%) $\lim_{x\to-\infty}(\sqrt{x^2+x}+x)$

(b) (8%) $\lim_{x\to0}\left(\frac{\sin x}{x}\right)^{\frac{1}{1-\cos x}}$

(c) (8%) $\lim_{x\to\infty}\frac{\sin\left(\frac{1}{\sqrt{x^2+1}}\right)}{\sqrt{x^2+2}-\sqrt{x^2-1}}$

(d) (8%) $\lim_{x\to\infty}\left[\left(\frac{x}{1+x}\right)^x-\frac{1}{e}\right]x$

Solution:

(a)

$$\begin{aligned}
\lim_{x\to-\infty}(\sqrt{x^2+x}+x) &= \lim_{x\to-\infty}(\sqrt{x^2+x}+x)\frac{\sqrt{x^2+x}-x}{\sqrt{x^2+x}-x} && \mathbf{(3)}\\
&= \lim_{x\to-\infty}\frac{x}{\sqrt{x^2+x}-x}\\
&= \lim_{x\to-\infty}\frac{1}{\frac{-1}{-x}\sqrt{x^2+x}-1} \quad (\text{as } x<0) && \mathbf{(5)}\\
&= \lim_{x\to-\infty}\frac{1}{-1\sqrt{\frac{x^2+x}{x^2}}-1}\\
&= \lim_{x\to-\infty}\frac{1}{-\sqrt{1+\frac{1}{x}}-1} = -\frac{1}{2} && \mathbf{(8)}
\end{aligned}$$

(b) Since

$$\lim_{x\to0}\frac{\sin x}{x}=1,$$

we see that

$$\begin{aligned}
&\lim_{x\to0}\frac{\sin(\frac{1}{\sqrt{x^2+1}})}{\sqrt{x^2+2}-\sqrt{x^2-1}}\\
&=\lim_{x\to0}\frac{\sin(\frac{1}{\sqrt{x^2+1}})}{\frac{1}{\sqrt{x^2+1}}}\cdot\lim_{x\to0}\frac{\sqrt{x^2+2}+\sqrt{x^2-1}}{\sqrt{x^2+1}\times3}\\
&\to1\times\frac{2}{3} && \mathbf{(+3)}
\end{aligned}$$

as $x\to\infty$. Therefore

$$\lim_{x\to\infty}\frac{\sin(\frac{1}{\sqrt{x^2+1}})}{\sqrt{x^2+2}-\sqrt{x^2-1}}=\frac{2}{3}. \qquad \mathbf{(+3)}$$

(c) Rewrite as

$$(\frac{\sin x}{x})^{\frac{1}{1-\cos x}}=\exp(\frac{1}{1-\cos x}\ln|\frac{\sin x}{x}|).$$

By l'Hopital theorem, we get

$$\begin{aligned}
\lim_{x\to0}\frac{1}{1-\cos x}\ln|\frac{\sin x}{x}| &= \lim_{x\to0}\frac{\frac{x}{\sin x}\frac{x\cos x-\sin x}{x^2}}{\sin x}\\
&= \lim_{x\to0}\frac{x\cos x-\sin x}{x\sin^2 x} && \mathbf{(3)}\\
&= \lim_{x\to0}\frac{-x\sin x}{\sin^2 x+2x\sin x\cos x}\\
&= \lim_{x\to0}\frac{-x}{\sin x+2x\cos x}\\
&= \lim_{x\to0}\frac{-1}{3\cos x-2x\sin x}=-\frac{1}{3}. && \mathbf{(6)}
\end{aligned}$$

Since e^x is continuous **(7)**, we get

$$\lim_{x\to0}(\frac{\sin x}{x})^{\frac{1}{1-\cos x}}=e^{-1/3}. \qquad \mathbf{(8)}$$

(d)

$$\lim_{x\to\infty}\frac{\left(\frac{1}{1+\frac{1}{x}}\right)^{x}-\frac{1}{e}}{\frac{1}{x}}$$

$$=\lim_{t\to 0^+}\frac{\left(\frac{1}{1+t}\right)^{\frac{1}{t}}-\frac{1}{e}}{t} \qquad (x=\frac{1}{t}) \qquad \textbf{(+2)}$$

$$=\lim_{t\to 0^+}\left(\frac{1}{1+t}\right)^{\frac{1}{t}}\frac{-\frac{t}{1+t}+\ln(1+t)}{t^2} \qquad (L'Hopital)$$

$$=\frac{1}{e}\lim_{t\to 0^+}\frac{-t+(1+t)\ln(1+t)}{t^2(1+t)} \qquad (L'Hopital) \qquad \textbf{(+2)}$$

$$=\frac{1}{e}\lim_{t\to 0^+}\frac{\ln(1+t)}{2t+3t^2} \qquad (L'Hopital) \qquad \textbf{(+2)}$$

$$=\frac{1}{e}\lim_{t\to 0^+}\frac{\frac{1}{1+t}}{2+6t}=\frac{1}{2e} \qquad (L'Hopital) \qquad \textbf{(+2)}$$

2. (12%) Let $f(x) = \begin{cases} x^\alpha \sin\left(\dfrac{1}{x^\beta}\right), & x > 0 \\ 0, & x = 0 \\ \dfrac{\sin(x^\beta)}{1-\cos x}, & x < 0. \end{cases}$

(a) For what values of α and β will $f(x)$ be continuous at $x = 0$?

(b) For what values of α and β will $f(x)$ be differentiable at $x = 0$?

Solution:

(a) $f(x)$ is continuous at $x = 0$

$\implies 0 = f(0) = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x)$ (1 point)

(1) $f(0) = 0$

(2) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^\alpha \sin(\frac{1}{x^\beta})$ (1 point)

since

$$-1 \le \sin(\frac{1}{x^\beta}) \le 1 \ \forall \beta \in \mathbb{R}$$

and

$$\lim_{x\to 0^+} -x^\alpha = 0 = \lim_{x\to 0^+} x^\alpha \text{ if } \alpha > 0$$

hence (by Squeeze theorem) (1 point)

$$\lim_{x\to 0^+} f(x) = 0 \text{ ,if } \alpha > 0 \text{ (1 point)}$$

if $\alpha \le 0$ and $\beta < 0$, then

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x^\alpha \sin(\frac{1}{x^\beta}) = \lim_{x\to 0^+} \frac{\sin x^{-\beta}}{x^{-\beta}} x^{\alpha-\beta} = 0 \text{ if } \alpha - \beta > 0$$

(3)

$$\begin{aligned} \lim_{x\to 0^-} f(x) &= \lim_{x\to 0^-} \frac{\sin x^\beta}{1-\cos x} \\ &= \lim_{x\to 0^-} \frac{\beta x^{\beta-1}\cos x^\beta}{\sin x} \text{ (by l'Hospital's rule) (1 point)} \\ &= \lim_{x\to 0^-} \frac{x}{\sin x} \beta x^{\beta-2} \cos x^\beta = 0 \text{ if } \beta > 2 \ (1 point) \end{aligned}$$

note that if $\beta = 0$, then

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{\sin x^0}{1-\cos x} = \lim_{x\to 0^-} \frac{\sin 1}{1-\cos x}$$

it doesn't converge

Therefore , $f(x)$ is continuous at $x = 0$ if $\alpha > 0$ and $\beta > 2$

2.(b) $f(x)$ is differentiable at $x = 0$ iff $f'(0^+) = f'(0^-)$ (1 point)

Moreover, $f(x)$ is differentiable at $x = 0$

$\implies f(x)$ is continuous at $x = 0 \implies \alpha > 0$ and $\beta > 2$

(1)

$$\begin{aligned} f'(0^+) &= \lim_{h\to 0^+} \frac{f(h) - f(0)}{h - 0} = \lim_{h\to 0^+} \frac{h^\alpha \sin(\frac{1}{h^\beta}) - 0}{h} \\ &= \lim_{h\to 0^+} h^{\alpha-1} \sin\frac{1}{h^\beta} \ \text{1 point} \end{aligned}$$

since

$$-1 \le \sin\left(\frac{1}{h^{\beta}}\right) \le 1 \ \forall \beta \in \mathbb{R}$$

and

$$\lim_{h\to 0^+} -h^{\alpha-1} = 0 = \lim_{h\to 0^+} h^{\alpha-1} \text{ if } \alpha - 1 > 0$$

hence (by Squeeze theorem) (1 point)

$$f'(0^+) = 0 \text{ if } \alpha > 1 \text{ (1 point)}$$

if $\alpha - 1 \le 0$ and $\beta < 0$, then

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h - 0} = \lim_{h\to 0^+} \frac{\sin h^{-\beta}}{h^{-\beta}} h^{\alpha-\beta-1} = 0 \text{ if } \alpha - \beta - 1 > 0$$

(2)

$$\begin{aligned} f'(0^-) &= \lim_{h\to 0^-} \frac{f(h) - f(0)}{h - 0} = \lim_{h\to 0^-} \frac{\frac{\sin h^{\beta}}{1-\cos h}}{h} = \lim_{h\to 0^-} \frac{\sin h^{\beta}}{h^{\beta}} \frac{h^{\beta-1}}{1 - \cos h} \\ &= \lim_{h\to 0^-} \frac{h^{\beta-1}}{1 - \cos h} = \lim_{h\to 0^-} \frac{(\beta - 1)h^{\beta-2}}{\sin h} \text{ (by l'Hospital's rule)} T \\ &= \lim_{h\to 0^-} \frac{h}{\sin h} (\beta - 1) h^{\beta-3} = 0 \text{ if } \beta - 3 > 0 \text{ (1 point)} \end{aligned}$$

hence

$$f'(0^-) = 0 \text{ if } \beta > 3 \text{ (1 point)}$$

note that if $\beta = 1$, then

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h - 0}) = \lim_{h\to 0^-} \frac{1}{1 - \cos h}$$

it doesn't converge

Therefore , $f(x)$ is differentiable at $x = 0$ if $\alpha > 1$ and $\beta > 3$

3. (8%) Let $f(x)$ be a twice differentiable one-to-one function. Suppose that $f(2) = 1$, $f'(2) = 3$, $f''(2) = e$. Find $\frac{d}{dx}f^{-1}(1)$ and $\frac{d^2}{dx^2}f^{-1}(1)$.

Solution:
let $y = f(x)$, then $x = f^{-1}(y)$ and when $x = 2$, $y = 1$
(1)

$$\frac{dy}{dx}\frac{dx}{dy} = 1 \text{ (2 points)}$$

implies that

$$\frac{d}{dy}f^{-1}(y) = \frac{1}{f'(x)}$$

hence

$$\frac{d}{dy}f^{-1}(1) = \frac{1}{f'(2)} = \frac{1}{3} \text{ (2 points)}$$

(2)

$$\frac{d}{dx}\left(\frac{dy}{dx}\frac{dx}{dy}\right) = \frac{d}{dx}(1) \implies \frac{d^2y}{dx^2}\frac{dx}{dy} + \left(\frac{dy}{dx}\right)^2\frac{dx}{dy} = 0 \text{ (2 points.)}$$

i.e.

$$f''(2)\frac{d}{dy}f^{-1}(1) + (f'(2))^2\frac{d^2}{dy^2}f^{-1}(1) = 0$$

i.e.

$$e\frac{1}{3} + 3^2\frac{d^2}{dy^2}f^{-1}(1) = 0$$

i.e.

$$\frac{d^2}{dy^2}f^{-1}(1) = \frac{-e}{27} \text{ (2 points)}$$

[another way]
let $g(x) = f^{-1}(x)$

since $g(f(x)) = x$, we have $g'(f(x))f'(x) = 1$ (2 points)

that is ,

$$g'(1) = g'(f(2)) = \frac{1}{f'(2)} = \frac{1}{3} \text{ (2 points)}$$

moreover

$$\begin{aligned}
&\frac{d}{dx}[g'(f(x))f'(x)] = \frac{d}{dx}(1) \\
&\implies g''(f(x))[f'(x)]^2 + g'(f(x))f''(x) = 0 \text{ (2 points)} \\
&\implies g''(1) = \frac{-e}{3}\frac{1}{3^2} = \frac{-e}{27} \text{ (2 points)}
\end{aligned}$$

[another way]

$$\begin{aligned}
&f(f^{-1}(x)) = x \text{ (2 points)} \\
&\implies f'(f^{-1}(x))\frac{d}{dx}f^{-1}(x) = 1 \\
&\implies \frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))} \text{ (2 points)} \\
&\implies \frac{d^2}{dx^2}f^{-1}(x) = \frac{d}{dx}\frac{1}{f'(f^{-1}(x))} = \frac{-\frac{d}{dx}f^{-1}(x)}{f'(f^{-1}(x))^2} \text{ (2 points)} \\
&\implies \frac{d}{dx}f^{-1}(1) = \frac{1}{3} \text{ (2 points) and } \frac{d^2}{dx^2}f^{-1}(1) = \frac{-e}{27} \text{ (2 points)}
\end{aligned}$$

4. (8%) Find the value of the number c such that the families of curves $y = (x+\alpha)^{-1}$ and $y = c(x+\beta)^{1/3}$ are orthogonal trajectories, that is, every curve in one family is orthogonal to every curve in the other family.

Solution:

$y = (x+\alpha)^{-1}$ then $y' = \dfrac{-1}{(x+\alpha)^2}$ (1 pt)

$y = c(x+\beta)^{\frac{1}{3}}$ then $y' = \dfrac{c}{3}(x+\beta)^{\frac{-2}{3}}$ (1 pt)

Let point of intersect be (x_0, y_0)

Orthogonal $\Rightarrow \dfrac{-1}{(x_0+\alpha)^2} \cdot \dfrac{c}{3}(x_0+\beta)^{\frac{-2}{3}} = -1$

$\Rightarrow c = 3(x_0+\alpha)^2(x_0+\beta)^{\frac{2}{3}}$ (2 pts)

We also have $y_0 = \dfrac{1}{x_0+\alpha} = c(x+\beta)^{\frac{1}{3}}$

$\Rightarrow \dfrac{1}{c(x_0+\alpha)} = (x+\beta)^{\frac{1}{3}}$ (2 pts)

combine with the equation above we have

$c = \dfrac{3}{c^2} \Rightarrow c^3 = 3 \Rightarrow c = \sqrt[3]{3}$ (2 pts)

5. (8%) Find the nth derivative of the function $f(x) = \dfrac{x^n}{1-x}$.

Solution:

Here are two ways to compute $f^{(n)}(x)$.

First one need to write $f(x) = \dfrac{x^n}{1-x} = \dfrac{x^n - 1}{1-x} + \dfrac{1}{1-x}$

$= -(x^{n-1} + x^{n-2} + \cdots + 1) + \dfrac{1}{1-x}$ (3 pts)

Note first term become zero after n times of differentiation. (1 pt)

$(\dfrac{1}{1-x})' = (-1) \cdot \dfrac{1}{(1-x)^2} \cdot (-1) = \dfrac{1}{(1-x)^2}$

$(\dfrac{1}{1-x})^{(k)} = \dfrac{k!}{(1-x)^{k+1}}$

So we have $f^{(n)}(x) = \dfrac{n!}{(1-x)^{n+1}}$. (4 pts)

Second way is to apply Leibniz's rule.

$f(x) = \dfrac{x^n}{1-x} = x^n \cdot \dfrac{1}{1-x}$

Then $f^{(n)}(x) = \Sigma_{i=0}^{n}\ C_i^n\ (x^n)^{(n-i)} \cdot (\dfrac{1}{1-x})^{(n)}$ (4 pts)

$= \Sigma_{i=0}^{n}\ C_i^n\ \dfrac{n!}{i!} x^i \cdot (i!) \dfrac{1}{(1-x)^{(i+1)}}$

$= \Sigma_{i=0}^{n}\ C_i^n\ n! \dfrac{x^i}{(1-x)^{i+1}}$ (4 pts)

6. (8%) Suppose that three points on the parabola $y = x^2$ have the property that their normal lines intersect at a common point. Show that the sum of their x-coordinates is 0.

Solution:

Let (x_1, x_1^2), (x_2, x_2^2), (x_3, x_3^2) be such three points.
If $x_1x_2x_3 = 0$, say $x_3 = 0$. Then the common point is on the y-axis.
The normal lines passing (x_1, x_1^2), (x_2, x_2^2) are
$y - x_1^2 = \frac{-1}{2x_1}(x - x_1)$, $y - x_2^2 = \frac{-1}{2x_2}(x - x_2)$
$\Rightarrow 0 = x = -2x_1x_2(x_1 + x_2)$
Since $x_1x_2 \neq 0$, we have $x_1 + x_2 = 0$. Hence $x_1 + x_2 + x_3 = 0$.
Now if $x_1x_2x_3 \neq 0$, the normal lines passing (x_1, x_1^2), (x_2, x_2^2), (x_3, x_3^2) are
$y - x_1^2 = \frac{-1}{2x_1}(x - x_1)$, $y - x_2^2 = \frac{-1}{2x_2}(x - x_2)$, $y - x_3^2 = \frac{-1}{2x_2}(x - x_3)$
$\Rightarrow x = -2x_1x_2(x_1 + x_2) = -2x_2x_3(x_2 + x_3) = -2x_1x_3(x_1 + x_3)$
$\Rightarrow x_1(x_1 + x_2) = x_3(x_2 + x_3)$
$\Rightarrow x_1(x_1 + x_2 + x_3 - x_3) = x_3(x_2 + x_3 + x_1 - x_1)$
$\Rightarrow (x_1 - x_3)(x_1 + x_2 + x_3) = 0$. Hence $x_1 + x_2 + x_3 = 0$.

評分標準
寫出法線方程式並把點帶入得三分
解出交點的 x 座標得兩分
得到 x 座標相加是0得三分

7. (12%) A cone-shaped paper drinking cup is to be made to hold 9 cm^3 of water. Find the height and radius of the cup that will use the smallest amount of paper.

Solution:

We have $\frac{1}{3}\pi r^2 h = 9$, $\theta = \frac{2\pi r}{\sqrt{r^2+h^2}}$

$\Rightarrow A = \frac{1}{2}(\sqrt{r^2+h^2})^2 \frac{2\pi r}{\sqrt{r^2+h^2}} = \pi r\sqrt{r^2+h^2}$

So $A(r) = \pi r\sqrt{r^2+(\frac{27}{\pi r^2})^2} = \pi\sqrt{r^4+\frac{729}{\pi^2 r^2}}$

$\Rightarrow A'(r) = \pi(\frac{4r^3-\frac{1458}{\pi^2 r^3}}{2\sqrt{r^4+\frac{729}{\pi^2 r^2}}})$

Let $A'(r) = 0 \Rightarrow 4r^3 - \frac{1458}{\pi^2 r^3} = 0 \Rightarrow r = \frac{3}{\sqrt[6]{2\pi^2}}$

Then $h = \frac{27}{\pi(\frac{3}{\sqrt[6]{2\pi^2}})^2} = 3\sqrt[3]{\frac{2}{\pi}}$.

These are answer since for $r < \frac{3}{\sqrt[6]{2\pi^2}}$, $A'(r) < 0$ and for $r > \frac{3}{\sqrt[6]{2\pi^2}}$, $A'(r) > 0$.

評分標準
列出體積關係式得兩分
算出扇形角度以及半徑各得一分
列出所要求的面積式子得一分 換成同一個變數再一分
對面積式子微分找出 critical number 得兩分
求出 r 和 h 各一分
說明為何是極小值得兩分

8. (12%)

(a) Suppose that $f(x)$ and $g(x)$ are differentiable on open interval containing $[a,b]$ and $f(a) > g(a)$, $f(b) > g(b)$. Show that if the equation $f(x) = g(x)$ has exactly one solution on $[a,b]$ then at the solution $x_0 \in [a,b]$, $f(x)$ and $g(x)$ have the same tangent line.
(Hint: Consider $h(x) = f(x) - g(x)$. Show that $h(x) \geq 0$ for all $x \in [a,b]$.)

(b) For $\alpha > 0$, if the equation $e^x = kx^\alpha$ has exact one solution on $[0,\infty)$, solve k in terms of α.

Solution:

(a) $h(x) = f(x) - g(x)$ is diff on $[a,b]$. $h(a) > 0$, $h(b) > 0$
If $h(\bar{x}) < 0$ for some $\bar{x} \in (a,b)$ **((+2): Correct assumption to start with.)**,
then by the *intermediate value thm*, there are some $x_1 \in [a,\bar{x}]$ and $x_2 \in [\bar{x},b]$ s.t. $h(x_1) = 0 = h(x_2)$ i.e. $f(x) = g(x)$ has at least two solution $x_1, x_2 \in [a,b]$. **((+2): Use IVT.)**
$\therefore h(x) \geq 0 \ \forall \ x \in [a,b]$ if $f(x) = g(x)$ has exactly one solution on $[a,b]$ $\rightarrow\leftarrow$

Suppose that r_0 is the only root for $f(x) = g(x)$, $r_0 \in [a,b]$. Then $h(r_0)$ is a local minimum value. **((+2): See local minimum)**,

$\because h(x)$ is diff on $[a,b]$ $\therefore h'(r_0) = 0 \Rightarrow f'(r_0) = g'(r_0)$. **((+2): Use Rolle's Theorem to conclude.)**

(b) $f(x) = e^x$, $g(x) = kx^\alpha$.
for $x = 0$, $f(0) = 1 > g(0) = 0$
for x large enough $f(x) > g(x)$.
Hence if $e^x = kx^\alpha$ has exactly one solution on $[0,\infty)$ then at the root $x = x_0$, $f(x)$ and $g(x)$ have the same tangent line.
i.e. if $f(x_0) = g(x_0)$ then $f'(x_0) = g'(x_0)$. **((+2): Apply part (a))**,

$$\begin{cases} e^{x_0} = kx_0^\alpha & -(1) \\ e^{x_0} = k\alpha x_0^{\alpha-1} & -(2) \end{cases} \Rightarrow kx_0^\alpha = k\alpha x_0^{\alpha-1} \Rightarrow x_0 = \alpha$$

(1) $\Rightarrow e^\alpha = k\alpha^\alpha$, $k = (\frac{e}{\alpha})^\alpha$. **((+2): Find correct answer)**.

9. (20%) Let $f(x) = (x^3 + x^2)^{1/3}$.

(a) Find all asymptotes of $f(x)$.

(b) Find the intervals of increase or decrease.

(c) Find the intervals of concavity.

(d) Find the local maximum and minimum values.

(e) Find the inflection points.

(f) Sketch the graph of $y = f(x)$.

Solution:

(a) Since $f(x)$ is finite for any finite $x \in \mathbb{R}$ and $f(x) \to \pm\infty$ as $x \to \pm\infty$, it does not have any vertical or horizontal asymptotes. However, since

$$\lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} (1 + \frac{1}{x})^{\frac{1}{3}} = 1 \qquad (2\%)$$

and

$$\begin{aligned}\lim_{x\to\pm\infty} (f(x) - 1\cdot x) &= \lim_{x\to\pm\infty} \frac{(x^3+x^2)-x^3}{(x^3+x^2)^{\frac{2}{3}} + x(x^3+x^2)^{\frac{1}{3}} + x^2} \\ &= \lim_{x\to\pm\infty} \frac{1}{(1+\frac{1}{x})^{\frac{2}{3}} + (1+\frac{1}{x})^{\frac{1}{3}} + 1} \\ &= \frac{1}{3} \qquad (2\%)\end{aligned}$$

f has a slant asymptote $y = x + \dfrac{1}{3}$.

(b)

$$f(x) = (x^3 + x^2)^{\frac{1}{3}}$$

$$f'(x) = \frac{1}{3}(x^3+x^2)^{-\frac{2}{3}}(3x^2+2x) = \frac{3x+2}{3x^{\frac{1}{3}}(x+1)^{\frac{2}{3}}} \qquad (2\%)$$

$f'(x) > 0$ for $x \in (-\infty, -\frac{2}{3})$ or $(0,\infty)$, and $f'(x) < 0$ for $x \in (-\frac{2}{3}, 0)$.
$\Rightarrow f(x)$ is increasing on $(-\infty, -\frac{2}{3})$ and $(0,\infty)$, decreasing on $(-\frac{2}{3}, 0)$. (3%)

(c)

$$f''(x) = \frac{1}{3}[-\frac{2}{3}(x^3+x^2)^{-\frac{5}{3}}(3x^2+2x)^2 + (x^3+x^2)^{-\frac{2}{3}}(6x+2)] = -\frac{2}{9x^{\frac{4}{3}}(x+1)^{\frac{5}{3}}} \qquad (2\%)$$

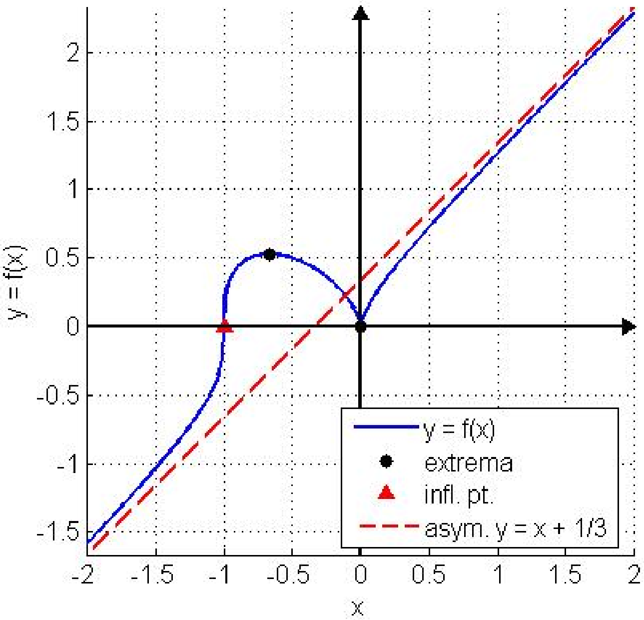
$f''(x) > 0$ for $x \in (-\infty, -1)$, and $f''(x) < 0$ for $x \in (-1, 0)$ or $(0,\infty)$.
$\Rightarrow f(x)$ is concave upward on $(-\infty,-1)$, concave downward on $(-1,0)$ and $(0,\infty)$. (3%)

(d) $f'(x)$ goes from positive to negative across $x = -\frac{2}{3}$ and from negative to positive across $x = 0$, and $f(x)$ is defined at these points.
$\Rightarrow f(-\frac{2}{3}) = \frac{\sqrt[3]{4}}{3}$ is the local maximum (1%) and $f(0) = 0$ is the local minimum (1%).

(e) $f''(x)$ changes sign only across $x = -1$ and f is continuous at that point.
$\Rightarrow f(-1) = 0$ is the only inflection point (1%).

(f)

x		-1		$-\frac{2}{3}$		0	
$f'(x)$	+	X	+	0	−	X	+
$f''(x)$	+	X	−		−	X	−
$f(x)$		0		$\frac{\sqrt[3]{4}}{3}$		0	



y = f(x)
extrema
infl. pt.
asym. y = x + 1/3
x
y = f(x)