微乙小考六 (2015/6/11)

1. (6%) 計算瑕積分 $\int_1^3 (x-2)^{-\frac{2}{5}} dx$.

   sol: The function $(x-2)^{-\frac{2}{5}}$ is not continuous at $x=2$. We must calculate this improper integral by

$$\int_1^3 (x-2)^{-\frac{2}{5}}\,dx = \int_1^{2^-} (x-2)^{-\frac{2}{5}}\,dx + \int_{2^+}^3 (x-2)^{-\frac{2}{5}}\,dx$$

   For the first term,

$$\begin{aligned}\int_1^{2^-} (x-2)^{-\frac{2}{5}}\,dx &= \lim_{t\to 2^-} \int_1^t (x-2)^{-\frac{2}{5}}\,dx \\ &= \lim_{t\to 2^-} \frac{5}{3}(x-2)^{\frac{3}{5}}\Big|_1^t \\ &= \lim_{t\to 2^-} \frac{5}{3}\left[(t-2)^{\frac{3}{5}} - (-1)^{\frac{3}{5}}\right] = \frac{5}{3}\end{aligned}$$

   Similarly,

$$\begin{aligned}\int_{2^+}^3 (x-2)^{-\frac{2}{5}}\,dx &= \lim_{t\to 2^+} \int_t^3 (x-2)^{-\frac{2}{5}}\,dx \\ &= \lim_{t\to 2^+} \frac{5}{3}(x-2)^{\frac{3}{5}}\Big|_t^3 \\ &= \lim_{t\to 2^+} \frac{5}{3}\left[(1)^{\frac{3}{5}} - (t-2)^{\frac{3}{5}}\right] = \frac{5}{3}\end{aligned}$$

   Thus,

$$\int_1^3 (x-2)^{-\frac{2}{5}}\,dx = \int_1^{2^-} (x-2)^{-\frac{2}{5}}\,dx + \int_{2^+}^3 (x-2)^{-\frac{2}{5}}\,dx = \frac{5}{3} + \frac{5}{3} = \frac{10}{3}$$

2. (7%) 令 $f_X(t) = \frac{1}{\sqrt{\pi}} e^{-(t-1)^2}$, $t \in R$. 求 $\mathrm{E}(X)$, $\mathrm{Var}(X)$.

   sol: The probability density function of X is $f_X(t) = \frac{1}{\sqrt{\pi}} e^{-(t-1)^2}$, $t \in R$.

   The expectation of X is

$$\begin{aligned}\mathrm{E}(X) &= \int_{-\infty}^{\infty} t\cdot f_X(t)\,dt = \int_{-\infty}^{\infty} t\cdot \frac{1}{\sqrt{\pi}} e^{-(t-1)^2}\,dt \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (u+1)\cdot e^{-u^2}\,du \quad [\,\text{Let } u = t-1\,] \\ &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u\cdot e^{-u^2}\,du + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du\end{aligned}$$

   Note that

$$\int_0^{\infty} u\cdot e^{-u^2}\,du = \lim_{t\to\infty} \int_0^t u\cdot e^{-u^2}\,du = \lim_{t\to\infty} -\frac{1}{2} e^{-u^2}\Big|_0^t = \lim_{t\to\infty} -\frac{1}{2}\left(e^{-t^2} - 1\right) = \frac{1}{2}$$

   Similarly, we can get

$$\int_{-\infty}^{0} u\cdot e^{-u^2}\,du = -\frac{1}{2}$$

Thus,

$$\mathrm{E}(X) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u \cdot e^{-u^2}\, du + \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\, du = \frac{1}{\sqrt{\pi}} \left(-\frac{1}{2} + \frac{1}{2} \right) + \frac{1}{\sqrt{\pi}} \cdot \sqrt{\pi} = 1$$

The variance of X is

$$\begin{aligned}
\mathrm{Var}(X) &= \mathrm{E}[(X - \mathrm{E}(X))^2] = \mathrm{E}[(X-1)^2] \\
&= \int_{-\infty}^{\infty} (t-1)^2 \cdot f_X(t)\, dt = \int_{-\infty}^{\infty} (t-1)^2 \cdot \frac{1}{\sqrt{\pi}} e^{-(t-1)^2}\, dt \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 \cdot e^{-u^2}\, du \quad [\,\text{Let } u = t-1\,] \\
&= \frac{1}{\sqrt{\pi}} \cdot \frac{-1}{2} \left(u e^{-u^2} \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-u^2} du \right) \quad [\,\text{Integration by parts }\,] \\
&= \frac{-1}{2\sqrt{\pi}} (-\sqrt{\pi}) = \frac{1}{2} \quad [\,\lim_{u\to\infty} u e^{-u^2} = 0 \text{ can be checked by L'Hopital Rule}\,]
\end{aligned}$$

3. (7%) 令 $f_X(t) = f_Y(t) = \frac{1}{\sqrt{\pi}} e^{-t^2}$, $t \in R$, X, Y 獨立。若 $Z = X + Y$, 求 Z 的機率密度函數 $f_Z(t)$.

sol: Since X, Y are independent and $Z = X + Y$, the probability density function of Z is

$$\begin{aligned}
f_Z(t) &= \int_{-\infty}^{\infty} f_X(x) f_Y(t-x)\, dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-x^2} \cdot \frac{1}{\sqrt{\pi}} e^{-(t-x)^2}\, dx \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-(2x^2 - 2tx + t^2)}\, dx \\
&= \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-2(x-\frac{t}{2})^2 - \frac{t^2}{2}}\, dx \\
&= \frac{1}{\pi} e^{-\frac{t^2}{2}} \int_{-\infty}^{\infty} e^{-u^2} \frac{1}{\sqrt{2}} du \quad [\,\text{Let } u = \sqrt{2}(x - \frac{t}{2})\,] \\
&= \frac{1}{\pi} e^{-\frac{t^2}{2}} \frac{1}{\sqrt{2}} \sqrt{\pi} = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \quad , t \in R
\end{aligned}$$