微乙小考三 (2015/4/16)

1. (6%) 求 $\int_0^1 \int_{y^2}^1 \left(\frac{\sin x}{\sqrt{x}} dx\right) dy$.

   sol:

$$\begin{aligned}\int_0^1 \int_{y^2}^1 \frac{\sin x}{\sqrt{x}} dxdy &= \int_0^1 \int_0^{\sqrt{x}} (\frac{\sin x}{\sqrt{x}} dy) dx \\ &= \int_0^1 \sin x dx \\ &= -\cos x|_0^1 \\ &= -\cos 1 + 1\end{aligned}$$

2. (7%) 令 Ω 是平面上 $\{2 \leq x^2 + y^2 \leq 18\}$ 和 $\{y \leq 0\}$ 的交集。求 $\int\int_\Omega e^{\frac{1}{2}(x^2+y^2)} dA$.

   sol:

$$\begin{aligned}\int\int_\Omega e^{\frac{1}{2}(x^2+y^2)} dA &= \int_\pi^{2\pi} \int_{\sqrt{2}}^{3\sqrt{2}} e^{\frac{1}{2}r^2} r dr d\theta \\ &= \int_\pi^{2\pi} 1 d\theta \cdot \int_{\sqrt{2}}^{3\sqrt{2}} e^{\frac{1}{2}r^2} r dr \\ &= \pi \cdot e^{\frac{1}{2}r^2}|_{\sqrt{2}}^{3\sqrt{2}} \\ &= \pi(e^9 - e)\end{aligned}$$

3. (7%) 在 $4x^2 + y^2 = 1$ 條件下，求 $f(x, y) = y^3 - 12x^2 y$ 的最大及最小值。

   sol: By Lagrange multiplier,

$$\begin{cases} -24xy = 8\lambda x & (1) \\ 3y^2 - 12x^2 = 2\lambda y & (2) \\ 4x^2 + y^2 = 1 & (3) \end{cases}$$

   by (1),we have $x(3y + \lambda) = 0 \Rightarrow x = 0$ or $\lambda = -3y$.

   If $x = 0$,then $y = \pm 1$.

   If $\lambda = -3y$, then we have $3y^2 = 4x^2$ by (2).
   We subtitute $4x^2 = 3y^2$ into (3), and then we have $y = \pm\frac{1}{2}$ and $x = \pm\frac{\sqrt{3}}{4}$.

   It is easy to compute that the maximum of f(x,y) is 1 at $(0, 1), (\pm\frac{\sqrt{3}}{4}, \frac{-1}{2})$, and the minimum of f(x,y) is -1 at $(0, -1), (\pm\frac{\sqrt{3}}{4}, \frac{1}{2})$.