1032微乙01-05班期中考解答和評分標準

1. (14%) 求函數 $f(x,y)=xy$ 在曲線 $x^2+xy+y^2=1$ 上之最大值，最小值及其所在之點。

Solution:
使用Lagrange multiplier method:

$$\begin{cases} y=\lambda(2x+y) \\ x=\lambda(2y+x) \\ x^2+xy+y^2=1 \end{cases} \quad (4pts)$$

$$\Rightarrow \begin{cases} \lambda=\dfrac{1}{3}, & x=\pm\dfrac{\sqrt{3}}{3}, y=\pm\dfrac{\sqrt{3}}{3} \\ \lambda=-1, & x=\pm1, y=\mp1 \end{cases} \quad (4pts)$$

將所得各點代入$f(x,y)$後比較，得到:
$f(x,y)$在$(\pm\frac{\sqrt{3}}{3},\pm\frac{\sqrt{3}}{3})$有最大值$\frac{1}{3}$。$(3pts)$
$f(x,y)$在$(\pm1,\mp1)$有最小值-1。$(3pts)$
(若過程完全錯誤，這部分不給分。)

2. (12%) 令 $f(x,y)=x^3-3\lambda xy+y^3$ 其中 λ 為實數，$\lambda\neq 0$. 求出 f 之候選點，並判斷其極值性質。(注意：和 λ 之值有關)

Solution:
$\nabla f(x,y)=(3x^2-3\lambda y, 3y^2-3\lambda x)=(0,0)$ (4 pts)
Now, we solve the following equations,

$$\begin{cases} x^2-\lambda y=0-(1) \\ y^2-\lambda x=0-(2) \end{cases}$$

From (1), we get $y=\dfrac{x^2}{\lambda}$, put into (2), we have

$$\frac{x^4}{\lambda^2}-\lambda x=0$$

Hence, $x=0$ or $x=\lambda$, so critical point: $(0,0),(\lambda,\lambda)$ (2 pts for each)
$D(x,y)=\begin{vmatrix} 6x & -3\lambda \\ -3\lambda & 6y \end{vmatrix}=36xy-9\lambda^2$ (2 pts)

1. $D(0,0)=-9\lambda^2<0\to$ (0,0) is saddle pt. (1 pt)
2. $D(\lambda,\lambda)=27\lambda^2>0$ & $f_{xx}(\lambda,\lambda)=6\lambda$, then
   (i) if $\lambda>0$, $f(\lambda,\lambda)=-\lambda^3$ is local min.
   (ii) if $\lambda<0$, $f(\lambda,\lambda)=-\lambda^3$ is local max. (1 pt)

3. (12%) 計算 $\displaystyle\iint_\Omega \frac{x^2}{x^2+y^2}dA$，其中 $\Omega: 1\le x^2+y^2\le 2$.

Solution:
Let $x=r\cos\theta$, $y=r\sin\theta$, then

$$\int\int_\Omega \frac{x^2}{x^2+y^2}dA=\int_0^{2\pi}\int_1^{\sqrt{2}} r\cos^2\theta drd\theta=\frac{1}{2}\int_0^{2\pi}\cos^2\theta d\theta=\frac{1}{4}\int_0^{2\pi}(\cos 2\theta+1)d\theta=\frac{\pi}{2}$$

* The score will be given based on **1**. domain of the integral, **2**. correctness of the integration. In general 3, 7 or 12 points will be given, some minor adjustments might occur if computational mistakes happened.

4. (12%) 令 Ω 為直線 $y-x=1$, $y-x=2$, $2x+y=0$, $2x+y=2$ 所圍成的區域。計算 $\iint_\Omega (y-x)(2x+y)dA$.

Solution:

By change of variables, let:

$$u = y - x, v = 2x + y$$
$$x = \frac{v-u}{3}, y = \frac{2u+v}{3}$$

So the integration region changes from:

$$y - x = 1$$
$$y - x = 2$$
$$2x + y = 0$$
$$2x + y = 2$$

to:

$$u = 1$$
$$u = 2$$
$$v = 0$$
$$v = 2$$

The Jacobian from change of coordinate (x, y) to (u, v) is:

$$J = \begin{bmatrix} \frac{\partial x}{\partial u} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} \frac{\partial y}{\partial v} \end{bmatrix} = \begin{bmatrix} \frac{-1}{3} \frac{1}{3} \\ \frac{2}{3} \frac{1}{3} \end{bmatrix} = \frac{-1}{3}$$

The integration becomes:

$$\int\int (y-x)(2x+y)dA = \int_0^2 \int_1^2 uv|J|dudv = \frac{1}{3}\int_1^2 udu \int_0^2 vdv$$
$$= \frac{1}{3} * (\frac{u^2}{2})|_1^2 (\frac{v^2}{2})|_0^2 = 1$$

評分標準:

1: 變數變換並得到正確的 Jacobian: (3分)
2: 找出在變數變換後正確的積分範圍: (3分)
3: 雙重積分做對: (6分)

在過程中有小錯誤每次一分扣

常見的錯誤像是 Jacobian 沒有加絕對值都是扣一分

另外有同學直接用直角坐標去做,儘管圍城區域有畫對, 但通常積分範圍就開始不正確了, 所以只給1-2分

5. (12%) 設 $f(x, y)$ 為可微函數，且 $\frac{\partial f}{\partial x}(2,-2) = \sqrt{2}$，$\frac{\partial f}{\partial y}(2,-2) = \sqrt{5}$。令 $x = u - v$，$y = v - u$. 求 $\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v}$ 在 $u = 1$, $v = -1$ 之值。

Solution:

當$(u, v) = (1, -1)$時，$(x, y) = (2, -2)$ (1%)

$$\frac{\partial f}{\partial x}(2,-2) = \sqrt{2}, \frac{\partial f}{\partial y}(2,-2) = \sqrt{5} \quad (3\%)$$

$$\frac{\partial x}{\partial u} = 1, \frac{\partial x}{\partial v} = -1, \frac{\partial y}{\partial u} = -1, \frac{\partial y}{\partial v} = 1 \quad (3\%)$$

$$\begin{aligned}(\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v})\,|_{(u,v)=(1,-1)} &= (\frac{\partial f}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v})\,|_{(x,y)=(2,-2)} \quad (5\%)\\ &= \sqrt{2}\times 1 + \sqrt{5}\times(-1) + \sqrt{2}\times(-1) + \sqrt{5}\times 1 = 0\end{aligned}$$

註1：因為$(u,v)=(1,-1)$時，$(x,y)=(2,-2)$，之後的偏微分才可以直接帶入$\sqrt{2},\sqrt{5}$，所以一定要明確表示這一點
註2：如果不把點帶入，直接用代數，如下:

$$\frac{\partial f}{\partial u} + \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\times 1 + \frac{\partial f}{\partial y}\times(-1) + \frac{\partial f}{\partial x}\times(-1) + \frac{\partial f}{\partial y}\times 1 = 0$$

就不需要敘述(u,v)與(x,y)之間的關係，可以直接拿到滿分

6. (14%) 令 $f(x,y) = e^x\cos y + a\sin y$，其中 a 為一常數。
   (a) (7%) 求曲線 $f(x,y) = -1$ 在點 $(0,\pi)$ 之切線方程式，以 a 表示之。
   (b) (7%) 設方向導數 $\frac{\partial f}{\partial \vec{u}}(0,0)$ 之最大值發生在 $\vec{u} = \left(\frac{3}{5},\frac{4}{5}\right)$，求 a 之值。

Solution:

(a) 令$g(x,y) = e^x\cos y + a\sin y + 1 = 0$，則:

$$\nabla g(x,y) = (e^x\cos y, -e^x\sin y + a\cos y)$$

(3 分)

$$\nabla g(0,\pi) = (-1,-a)$$

(2 分)

切線方程為:

$$0 = \nabla g(0,\pi)\cdot(x-0, y-\pi) = -x - ay - a\pi$$

(2 分)

(b) 因最大值發生的方向為∇f，故$\vec{u} = (\frac{3}{5},\frac{4}{5})$與$\nabla f$平行:

$$\nabla f \parallel \vec{u}$$

(5 分)

即:

$$\frac{-1}{-a} = \frac{\frac{3}{5}}{\frac{4}{5}}$$
$$a = \frac{4}{3}$$

(2 分)

7. (12%) 計算下列積分值: $\iint\limits_{\Omega} \frac{3x^2}{(x^3+y^2)^2}dA$ 其中 $\Omega = [0,1]\times[1,3]$。

Solution:

$$\begin{aligned}\int_1^3\int_0^1 \frac{3x^2}{(x^3+y^2)^2}dxdy &= \int_1^3(\frac{1}{y^2} - \frac{1}{1+y^2})dy\\ &= (-\frac{1}{y} - \tan^{-1}y)|_1^3 = \frac{2}{3} - \tan^{-1}3 + \frac{\pi}{4}\end{aligned}$$

8. (12%) 計算下列積分值: $\iint\limits_R \frac{xe^y}{y} dA$ 其中 $R: 0 \leq x \leq 1,\ x^2 \leq y \leq x$ 所圍成的有限區塊。

Solution:

$$R = \{(x,y) \mid 0 \leq x \leq 1,\ x^2 \leq y \leq x\} = \{(x,y) \mid 0 \leq y \leq 1,\ y \leq x \leq \sqrt{y}\}$$

$$\begin{aligned}
\therefore \iint_R \frac{xe^y}{y} dA &= \int_0^1 \int_y^{\sqrt{y}} \frac{xe^y}{y} dx dy \qquad (7\%) \\
&= \int_0^1 \frac{e^y}{y} \cdot \frac{x^2}{2}\Big|_{x=y}^{\sqrt{y}} dy \\
&= \int_0^1 \frac{e^y}{y} \cdot \frac{1}{2}(y - y^2) dy \qquad (2\%) \\
&= \frac{1}{2}\left[\int_0^1 e^y dy - \int_0^1 ye^y dy\right] = \frac{1}{2}\left[e^y - (ye^y - e^y)\right]\Big|_0^1 \\
&= \frac{1}{2}e - 1 \qquad (3\%)
\end{aligned}$$