Chapter 9 Problems Plus

EX.1

we first use the Fundamental Theorem of Calculus to differentiate the given equation : 
\[ |f(x)|^2 = 100 + \int_0^x |f(t)|^2 + |f'(t)|^2 \, dt \Rightarrow 2f(x)f'(x) = |f(x)|^2 + |f'(x)|^2 \Rightarrow |f(x)|^2 + |f'(x)|^2 - 2f(x)f'(x) = 0 \Rightarrow 
[f(x) - f'(x)]^2 = 0 \Rightarrow f(x) = f'(x) \Rightarrow f(x) = Ce^x .\] 
Now \( f(0) = C = \pm 10 \) since \( |f(0)|^2 = 100 \). Hence \( f(x) = \pm 10e^x \)

EX.3

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h \to 0} \frac{f(x)f(h)-1}{h} = f(x) \lim_{h \to 0} \frac{f(h)-1}{h} = f(x)f'(0) = f(x) \Rightarrow f'(x) = f(x) \Rightarrow f(x) = e^x \text{ since } f(0) = 1 \]

EX.9

(a) While running from \((L,0)\) to \((x,y)\), the dog travels a distance
\[ s = \int_L^x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx = - \int_L^x \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx, \]
so \( \frac{ds}{dx} = -\sqrt{1 + \left( \frac{dy}{dx} \right)^2} \). The dog and rabbit run at the same speed, so the rabbit is at \((0,s)\) when the dog has traveled a distance \( s \). Then \( \frac{dy}{dx} = \frac{\sqrt{1 + z^2}}{L} \), since the dog run straight for the rabbit. Then
\[ s = y - x \frac{dy}{dx} = \frac{ds}{dx} = \frac{dy}{dx} = - \left( x \frac{d^2y}{dx^2} + 1 \frac{dy}{dx} \right) = -x \frac{d^2y}{dx^2} = \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \]

(b) Letting \( z = \frac{dy}{dx} \) then we have \( \frac{dz}{\sqrt{1+z^2}} = \frac{dx}{z} \), then integrate it, we have
\( \ln x = \int \frac{dz}{\sqrt{1+z^2}} = \ln (z + \sqrt{1+z^2}) + C \). Then \( C = \ln L \) since when \( x = L, z = 0 \). So \( \ln x = \ln (\sqrt{1+z^2} + z) + \ln L = \ln L(\sqrt{1+z^2} + z) \Rightarrow x = L(\sqrt{1+z^2} + z) \Rightarrow \sqrt{1+z^2} = \frac{x}{L} - z \Rightarrow 1 + z^2 = \left( \frac{x}{L} \right)^2 - 2z + z^2 \Rightarrow \left( \frac{x}{L} \right)^2 - 2z = 1 \Rightarrow z = \frac{x^2}{2L} - \frac{L}{2} \). Then \( y = \frac{x^2}{4L} - \frac{L \ln x}{2} + \frac{L \ln L}{2} - \frac{L}{4} = \frac{x^2-L^2}{4L} - \frac{L \ln x}{2} \)

(c) As \( x \to 0^+, y \to \infty \), so the dog never catches the rabbit.

EX.12

Let \( P(a,b) \) be any first-quadrant point on \( y = f(x) \). Then the tangent line at \( P \) is \( y = mx + b - ma \) where \( m = f'(a) \). If \( Q(0,c) \) is the \( y \)-intercept, then \( c = b - am \). If \( R(k,0) \) is the \( x \)-intercept, then
\[ k = \frac{am-b}{m} = a - \frac{b}{m} \]. We know that \( (a-0)^2 + [b-(b-am)]^2 = \sqrt{[a-(a - \frac{b}{m})]^2 + (b - 0)^2} \Rightarrow \]
\[ a^2 + a^2m^2 = \frac{b^2}{m^2} + b^2 \Rightarrow a^2m^2 + a^2m^4 = b^2 + b^2m^2 \Rightarrow a^2m^4 + (a^2 - b^2)m^2 - b^2 = 0 \Rightarrow (a^2m^2 - b^2)(m^2 + 1) = 0 \Rightarrow m^2 = \frac{b^2}{a^2} \], then \( m = -\frac{b}{a} \) since \( m \) is negative and \( a, b \) are positive. Then solve 

\[ \frac{dy}{dx} = -\frac{y}{x} \Rightarrow \frac{dy}{y} = -\frac{dx}{x} \Rightarrow \ln y = -\ln x + C \Rightarrow y = \frac{A}{x}. \]

Then \( A = 6 \) since \((3, 2)\) is on it.

**EX.14**

Let \( P(x_0, y_0) \) be a point on the curve, then the normal line has slope \( \frac{y_0}{2x_0} \). So the tangent line has slope \( -\frac{2x_0}{y_0} \). This gives the equation \( y' = -\frac{2x}{y} \Rightarrow ydy = -2xdx \Rightarrow \int ydy = \int (2x)dx \Rightarrow \frac{1}{2}y^2 = -x^2 + C \Rightarrow x^2 + \frac{1}{2}y^2 = C \). This is a family of ellipses.