1. (a) \( \frac{dP}{dt} = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100). \) Comparing to Equation 4, 
\( \frac{dP}{dt} = kP(1 - P/M), \) we see that the carrying capacity is \( M = 100 \) and the value of \( k \) is 0.05.
(b) The slopes close to 0 occur where \( P \) is near 0 or 100. The largest slopes appear to be on the line \( P = 50 \). The solutions are increasing for \( 0 < P_0 < 100 \) and decreasing for \( P_0 > 100 \).
(c) All of the solutions approach \( P = 100 \) as \( t \) increases. As in part (b), the solutions differ since for \( 0 < P_0 < 100 \) they are increasing, and for \( P_0 > 100 \) they are decreasing. Also, some have an IP and some don’t. It appears that the solutions which have \( P_0 = 20 \) and \( P_0 = 40 \) have inflection points at \( P = 50 \).
(d) The equilibrium solutions are \( P = 0 \) (trivial solution) and \( P = 100 \). The increasing solutions move away from \( P = 0 \) and all nonzero solutions approach \( P = 100 \) as \( t \to \infty \).

9. (a) Our assumption is that \( \frac{dy}{dt} = ky(1 - y) \), where \( y \) is the fraction of the population that has heard the rumor.
(b) Using the logistic equation (4), \( \frac{dP}{dt} = kP \left(1 - \frac{P}{M}\right) \), we substitute \( y = \frac{P}{M} \), \( P = My \), and \( \frac{dP}{dt} = M \frac{dy}{dt} \), to obtain \( M \frac{dy}{dt} = k(My)(1 - y) \) \( \iff \frac{dy}{dt} = ky(1-y) \), our equation in part (a).

Now the solution to (4) is \( P(t) = \frac{M}{1 + Ae^{-kt}} \), where \( A = \frac{M-P_0}{P_0} \).

We use the same substitution to obtain \( My = \frac{M}{1 + \frac{M-P_0}{My}e^{-kt}} \) \( \Rightarrow \ y = \frac{y_0}{y_0+(1-y_0)e^{-kt}} \).

Alternatively, we could use the same steps as outlined in the solution of Equation 4.
(c) Let \( t \) be the number of hours since 8 AM. Then \( y_0 = y(0) = \frac{80}{1000} = 0.08 \) and \( y(4) = \frac{1}{2} \), so \( \frac{1}{2} = y(4) = \frac{0.08}{0.08 + 0.92e^{-4k}}. \)

Thus, \( 0.08 + 0.92e^{-4k} = 0.16 \), \( e^{-4k} = \frac{0.2}{0.23} = \frac{2}{23} \), and \( e^{-k} = \left(\frac{2}{23}\right)^{1/4} \), so \( y = \frac{2}{2 + 23\left(\frac{2}{23}\right)^{t/4}}. \)

Solving this equation for \( t \), we get \( t = 4 \left[1 + \frac{\ln((1-y)/y)}{\ln \frac{2}{23}}\right]. \)

When \( y = 0.9 \), so \( t = 4 \left(1 - \frac{\ln 0.9}{\ln \frac{2}{23}}\right) \approx 7.6h \) or 7h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 PM.

10.
(a) \( P(0) = P_0 = 400 \), \( P(1) = 1200 \) and \( M = 10000 \). From the solution to the logistic differential equation \( P(t) = \frac{P_0 M}{P_0 + (M - P_0)e^{-kt}} \), we get \( P = \frac{400 \times 10000}{400 + 9600e^{-kt}} = \frac{10000}{1 + 24e^{-kt}}. \)

\[ P(1) = 1200 \implies 1 + 24e^{-k} = \frac{100}{12} \implies e^k = \frac{288}{88} = \frac{36}{11} \implies k = \ln \frac{36}{11}. \]

So

\[ P = \frac{10000}{1 + 24 \times \left(\frac{11}{36}\right)^t}. \]

(b) \[ 5000 = \frac{10000}{1 + 24 \times \left(\frac{11}{36}\right)^t} \implies t = \frac{\ln 24}{\ln 36 - \ln 11} \approx 2.68 \text{ years}. \]

15.

(a) \( \frac{dP}{dt} = kP - m = k \left( P - \frac{m}{k} \right) \). Let \( y = P - \frac{m}{k} \), so \( \frac{dy}{dt} = \frac{dP}{dt} \) and the differential equation becomes \( \frac{dy}{dt} = kt. \)

(b) Since \( k > 0 \), there will be an exponential expansion \( \iff P_0 - \frac{m}{k} > 0 \iff m < kP_0. \)

(c) The population will be constant if \( P_0 - \frac{m}{k} = 0 \iff m = kP_0 \). It will decline if \( P_0 - \frac{m}{k} < 0 \iff m > kP_0. \)

(d) \( P_0 = 8000000, \ k = \alpha - \beta = 0.016, \ m = 2100000 \implies m > kP_0 (= 128000) \), so by part (c), the population was declining.