2. \( y = \pm \sqrt{x^3 - x^4} \) \( \Rightarrow \) The loop of the curve is symmetric about \( y = 0 \), and therefore \( \bar{y} = 0 \). At each point \( x \) where \( 0 \leq x < 1 \), the lamina has a vertical length of \( \sqrt{x^3 - x^4} - (-\sqrt{x^3 - x^4}) = 2\sqrt{x^3 - x^4} \). Therefore,

\[
\bar{x} = \frac{\int_0^1 x \cdot 2\sqrt{x^3 - x^4} \, dx}{\int_0^1 2\sqrt{x^3 - x^4} \, dx} = \frac{\int_0^1 x\sqrt{x^3 - x^4} \, dx}{\int_0^1 \sqrt{x^3 - x^4} \, dx}.
\]

We evaluate the integrals separately:

\[
\int_0^1 x\sqrt{x^3 - x^4} \, dx = \int_0^1 x^{5/2} \sqrt{1-x} \, dx
\]

\[
= \left[ 2 \sin^6 \theta \cos \theta \sqrt{1-\sin^2 \theta} \, d\theta \right]_0^{\pi/2} \quad \left[ \sin \theta = \sqrt{x}, \cos \theta \, d\theta = \frac{dx}{2\sqrt{x}} \right]
\]

\[
= \left[ 2 \sin^6 \theta \cos^2 \theta \, d\theta \right]_0^{\pi/2}
\]

\[
= \left[ 2 \left( \frac{1}{2} (1 - \cos 2\theta) \right)^3 \frac{1}{2} (1 + \cos 2\theta) \, d\theta \right]_0^{\pi/2}
\]

\[
= \left[ \frac{1}{8} \left( 1 - 2\cos 2\theta + 2\cos 2\theta (1 - \sin^2 2\theta) - \frac{1}{4} (1 + \cos 4\theta)^2 \right) \right]_0^{\pi/2} \, d\theta
\]

\[
= \left[ \frac{1}{8} \left( \theta - \frac{1}{3} \sin 2\theta \right) \right]_0^{\pi/2} - \frac{1}{32} \left[ \frac{1}{8} \left( 1 + 2\cos 4\theta + \cos^2 4\theta \right) \right]_0^{\pi/2} \, d\theta
\]

\[
= \frac{\pi}{16} - \frac{1}{32} \left[ \theta + \frac{1}{2} \sin 4\theta \right]_0^{\pi/2} - \frac{1}{64} \left[ (1 + \cos 8\theta) \right]_0^{\pi/2} \, d\theta
\]
\[
\frac{5\pi}{128}
\]
\[
\int_{0}^{1} \sqrt{x^3 - x^4} \, dx = \int_{0}^{1} x^{3/2} \sqrt{1 - x} \, dx
\]
\[
= \int_{0}^{\pi/2} 2 \sin^4 \theta \cos \theta \sqrt{1 - \sin^2 \theta} \, d\theta
\]
\[
= \int_{0}^{\pi/2} 2 \sin^4 \theta \cos^2 \theta \, d\theta
\]
\[
= \int_{0}^{\pi/2} 2 \cdot \frac{1}{4} (1 - \cos 2\theta)^2 \cdot \frac{1}{2} (1 + \cos 2\theta) \, d\theta
\]
\[
= \int_{0}^{\pi/2} \frac{1}{4} (1 - \cos 2\theta - \cos^2 2\theta + \cos^3 2\theta) \, d\theta
\]
\[
= \int_{0}^{\pi/2} \frac{1}{4} \left[ 1 - \cos 2\theta - \frac{1}{2} (1 + \cos 4\theta) + \cos 2\theta (1 - \sin^2 2\theta) \right] \, d\theta
\]
\[
= \frac{1}{4} \left[ \frac{\pi}{2} - \frac{1}{8} \sin 4\theta - \frac{1}{6} \sin^3 2\theta \right]_0^\pi
\]
\[
= \frac{\pi}{16}
\]

Therefore, \( \bar{x} = \frac{5}{8} \), and \( (\bar{x}, \bar{y}) = \left( \frac{5}{8}, 0 \right) \).

3.

(a) The two spherical zones, whose surface areas we will call \( S_1 \) and \( S_2 \), are generated by rotation about \( y \)-axis of circular arcs, as indicated in the figure. The arcs are the upper and lower portions of the circle \( x^2 + y^2 = r^2 \) that are obtained when the circle is cut with the line \( y = d \). The portion of the upper arc in the first quadrant is sufficient to generate the upper spherical zone. That portion of the arc can be described by the relation \( x = \sqrt{r^2 - y^2} \) for \( d \leq y \leq r \). Thus, \( \frac{dx}{dy} = -y/\sqrt{r^2 - y^2} \) and

\[
ds = \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \sqrt{1 + \frac{y^2}{r^2 - y^2}} \, dy = \frac{r \, dy}{\sqrt{r^2 - y^2}}
\]

From Formula 8.2.8 we have

\[
S_1 = \int_{d}^{r} 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \int_{d}^{r} 2\pi \sqrt{r^2 - y^2} \, \frac{r \, dy}{\sqrt{r^2 - y^2}} = \int_{d}^{r} 2\pi r \, dy = 2\pi r (r - d)
\]

Similarly, we can compute \( S_2 = \int_{-r}^{d} 2\pi x \sqrt{1 + \left( \frac{dx}{dy} \right)^2} \, dy = \int_{-r}^{d} 2\pi r \, dy = 2\pi r (r + d) \). Note that \( S_1 + \)
\( S_2 = 4\pi r^2 \), the surface area of entire sphere.

(b) \( r = 6370 \text{ km} \) and \( d = r(\sin 75^\circ) \approx 6153 \text{ km} \),

so the surface area of the Arctic Ocean is about \( 2\pi r(r - d) \approx 2\pi (6370)(217) \approx 8.69 \times 10^6 \text{ km}^2 \).

(c) The area on the sphere lies between planes \( y = y_1 \) and \( y = y_2 \), where \( y_2 - y_1 = h \). Thus, we compute the surface area on the sphere to be

\[
S = \int_{y_1}^{y_2} 2\pi \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_{y_1}^{y_2} 2\pi r \, dy = 2\pi r(y_2 - y_1) = 2\pi rh.
\]

This equals the lateral area of a cylinder of radius \( r \) and height \( h \), since such a cylinder is obtained by rotating the line \( x = r \) about the \( y \)-axis, so the surface area of the cylinder between the planes \( y = y_1 \) and \( y = y_2 \) is

\[
A = \int_{y_1}^{y_2} 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy = \int_{y_1}^{y_2} 2\pi r \sqrt{1 + 0^2} \, dy = 2\pi r |^{y_2}_{y_1} = 2\pi r(y_2 - y_1) = 2\pi rh.
\]

6. The problem can be reduced to finding the line which minimizes the shaded area in the diagram. An equation of the circle in the first quadrant is \( x = \sqrt{1 - y^2} \). So the shaded area is

\[
A(h) = \int_0^h \left(1 - \sqrt{1 - y^2}\right) \, dy + \int_h^1 \sqrt{1 - y^2} \, dy
\]

\[
= \int_0^h (1 - \sqrt{1 - y^2}) \, dy - \int_1^h \sqrt{1 - y^2} \, dy
\]

\[
A'(h) = 1 - \sqrt{1 - h^2} - \sqrt{1 - h^2} \quad [\text{by FTC}] = 1 - 2\sqrt{1 - h^2}
\]

\[
A'(h) = 0 \iff \sqrt{1 - h^2} = \frac{1}{2} \quad \Rightarrow \quad 1 - h^2 = \frac{1}{4} \quad \Rightarrow \quad h^2 = \frac{3}{4} \quad \Rightarrow \quad h = \frac{\sqrt{3}}{2}
\]

\[
A''(h) = \frac{2h}{\sqrt{1 - h^2}} > 0, \text{ so } h = \frac{\sqrt{3}}{2} \text{ gives a minimum value of } A.
\]

Note: Another strategy is to use the angle \( \theta \) as the variable (see the diagram above) and show that \( A = \theta + \cos \theta - \frac{\pi}{4} - \frac{1}{2} \sin 2\theta \), which is minimized when \( \theta = \frac{\pi}{6} \).
\[ A_1 = 30 \quad \Rightarrow \quad \frac{1}{2} bh = 30 \quad \Rightarrow \quad bh = 60. \]

\[ \bar{x} = 6 \quad \Rightarrow \quad \frac{1}{A_2} \int_0^{10} xf(x) = 6 \quad \Rightarrow \quad \int_0^b x \left(\frac{h}{b} x + 10 - h\right) dx + \int_b^{10} x(10) dx = 6(70) \]

\[ \Rightarrow \int_0^b \left(\frac{h}{b} x^2 + 10x - hx\right) dx + 10 \cdot \frac{1}{2} [x^2]_b^{10} = 420 \]

\[ \Rightarrow \left[ \frac{h}{3b} x^3 + 5x^2 - \frac{h}{2} x^2 \right]_0^b + 5(100 - b^2) = 420 \]

\[ b = 8 \]

So

\[ h = \frac{15}{2} \]

and an equation of the line is

\[ y = \frac{15}{16} x + \frac{5}{2} \]

Now

\[ \bar{y} = \frac{1}{A_2} \int_0^{10} \frac{1}{2} [f(x)]^2 dx = \frac{1}{70 \times 2} \left[ \int_0^8 \left(\frac{15}{16} x + \frac{5}{2}\right)^2 dx + \int_8^{10} (10)^2 dx \right] = \frac{55}{14} \]
13. Solve for $y$: $x^2 + (x + y + 1)^2 = 1 \Rightarrow (x + y + 1)^2 = 1 - x^2 \Rightarrow x + y + 1 = \pm \sqrt{1 - x^2}$
\[y = -x - 1 \pm \sqrt{1 - x^2}\]

\[A = \int_{-1}^{1} \left[(-x - 1 + \sqrt{1 - x^2}) - (-x - 1 - \sqrt{1 - x^2})\right] dx = 2 \int_{-1}^{1} \sqrt{1 - x^2} dx = \pi \text{ [area of semicircle]}\]

\[\bar{x} = \frac{1}{A} \int_{-1}^{1} x \times 2\sqrt{1 - x^2} dx = 0 \text{ [odd function]}\]

\[\bar{y} = \frac{1}{A} \int_{-1}^{1} \frac{1}{2} \left[(-x - 1 + \sqrt{1 - x^2})^2 - (-x - 1 - \sqrt{1 - x^2})^2\right] dx\]

\[= -\frac{2}{\pi} \int_{-1}^{1} \left(x\sqrt{1 - x^2} + \sqrt{1 - x^2}\right) dx = -\frac{2}{\pi} \int_{-1}^{1} x\sqrt{1 - x^2} dx - \frac{2}{\pi} \int_{-1}^{1} \sqrt{1 - x^2} dx = -1\]

Thus, as expected, the centroid is $(\bar{x}, \bar{y}) = (0, -1)$. We might expect this result since the centroid of an ellipse is located at its center.