**Section 6.1 Areas Between Curves**

**EX.12**

![Graph showing the area between curves](image)

Solve $4x = 12 - y^2 = 4y$ to get $y = -6, 2$.

So the intersection points are $(-6, -6)$ and $(2, 2)$.

We write $4x + y^2 = 12$ as $x = 3 - y^2/4$.

In this, $x = 3 - y^2/4$ is on the right side and $x = y$ is on the left side.

Then the area is going to be

$$A = \int_{y=-6}^{y=2} (x_R - x_L)dy = \int_{-6}^{2} \left[ 3 - \frac{y^2}{4} - y \right] dy = \int_{-6}^{2} \left[ 3 - \frac{y^2}{4} - y \right] dy$$

$$= \left[ -\frac{y^3}{12} - \frac{y^2}{2} + 3y \right]_{-6}^{2} = \left( -\frac{2}{3} - 2 + 6 \right) - (18 - 18 - 18) = \frac{64}{3}$$

**EX.20**

![Graph showing a different area between curves](image)

First of all, we write $y = \sqrt{2-x}$ as $x = 2 - y^2 (y \leq 0)$.

In this, $x = y^4$ is on the left side and $x = 2 - y^2$ is on the right side.

They meet when $x = y^4 = (2-x)^2$. 
That is, \(x^2 - 5x + 4 = 0 \iff x = 1\) or \(4\) (not satisfying \(y = \sqrt{2 - x}\)).

So they must meet at \((1,1)\).

The area is

\[
A = \int_0^1 (x_R - x_L)dy = \int_0^1 [(2 - y^2) - y^4]dy = \int_0^1 (2 - y^2 - y^4)dy
\]

\[
= \left(2y - \frac{y^3}{3} - \frac{y^5}{5}\right)|_0^1 = 2 - \frac{1}{3} - \frac{1}{5}
\]

\[
= \frac{22}{15}
\]

**EX.21**

![Graph showing y = tan x and y = 2 sin x](image)

In this, we note that \(y = \tan x\) is below \(y = 2\sin x\) while \(x > 0\).

And it’s easy to check they meet at \(\pm(\pi/3, \sqrt{3})\).

The area is

\[
A = \int_{\pi/3}^{\pi} |2 \sin x - \tan x|dx
\]

By symmetry,

\[
A = 2 \int_0^{\pi/3} (2 \sin x - \tan x)dx = 2 (-2 \cos(x) - \ln |\sec(x)|)|_{0}^{\pi/3} = 2 \left[(-1 - \ln 2) - (-2 - 0)\right] = 2 - 2 \ln 2
\]

**EX.28**

We separate the area into two parts \(A\) and \(B\).

\(A\): Between \(y = x^2/4\) to \(y = 2x^2\), from \(x = 0\) to \(x = 1\). \(B\): Between \(y = x^2/4\) to \(y = 3 - x\), from \(x = 1\)
to $x = 2$. We have

$$A = \int_0^1 \left( 2x^2 - \frac{x^2}{4} \right) dx = \int_0^1 \frac{7x^2}{4} dx = \frac{7x^3}{12} \bigg|_0^1 = \frac{7}{12}$$

and

$$B = \int_1^2 \left[ 3 - x - \frac{x^2}{4} \right] dx = \int_1^2 3x - \frac{x^2}{2} - \frac{x^3}{12} \bigg|_1^2 = \left( 6 - 2 - \frac{2}{3} \right) - \left( 3 - \frac{1}{2} - \frac{1}{12} \right) = 1 - \frac{1}{12} = \frac{11}{12}.$$ 

Hence, the total area is

$$A + B = \frac{7}{12} + \frac{11}{12} = \frac{3}{2}.$$

**EX.42**

It’s routine to find the intersection points are $(1/2, \pm 1/2)$. For given $y$, $x$ is between $2y^2$ and $1 - |y|$. i.e. $2y^2 \leq x \leq 1 = |y|$ from $y = -1/2$ to $y = 1/2$. So the area is
\[ A = \int_{-\frac{1}{2}}^{\frac{1}{2}} [(1 - |y|) - (2y^2)] dy. \]

By symmetry,
\[ A = 2 \int_{0}^{\frac{1}{2}} [(1 - y - 2y^2)dy = 2 \left[ y - \frac{y^2}{2} - \frac{2y^3}{3} \right]_0^{\frac{1}{2}} = 2 \left[ \frac{1}{2} - \frac{1}{8} - \frac{1}{12} \right] = \frac{7}{12}. \]

**EX.49**

We find that \( y^2 = x^2(x+3) \) intersects itself at (0,0) and (-3,0).

Solve \( y \), we have \( y = \pm x\sqrt{x + 3} \).

Note that \( y = -x\sqrt{x + 3} \) is above \( y = x\sqrt{x + 3} \).

So the area is
\[ A = \int_{-3}^{0} \left[ (-x\sqrt{x + 3}) - (x\sqrt{x + 3}) \right] dx \]
\[ = -2 \int_{-3}^{0} x\sqrt{x + 3}dx. \]

Let \( u = \sqrt{x + 3} \) so we have \( x = u^2 - 3 \). Because \( dx = 2udu \),
\[ A = -2 \int_{u=0}^{u=\sqrt{3}} (2u^4 - 6u^2)du = -2 \left[ \frac{4u^5}{5} - 4u^3 \right]_0^{\sqrt{3}} \]
\[ = -\frac{36\sqrt{3}}{5} - 12\sqrt{3} = \frac{24\sqrt{3}}{5}. \]
EX.53

We want that \( \int_{-|c|}^{|c|} \left[ \left( c^2 - x^2 \right) - \left( x^2 - c^2 \right) \right] \, dx = 576. \)

\[
2 \int_{-|c|}^{|c|} \left( c^2 - x^2 \right) \, dx = 576
\]

\[
\left. \left( c^2 x - \frac{x^3}{3} \right) \right|_{-|c|}^{|c|} = 288
\]

\[
\left( c^2 |c| - \frac{|c|^3}{3} \right) - \left(-c^2 |c| + \frac{|c|^3}{3} \right) = 288
\]

\[
\frac{4|c|^3}{3} = 288
\]

\[
|c|^3 = 288 \times \frac{3}{4} = 216
\]

So \(|c| = 6\) and then \(c = \pm 6.\)

EX.55
Of course, \( y = mx \) and \( y = x/(x^2 + 1) \) intersect at \((0,0)\).

So we need to find another solution of \( mx = x/(x^2 + 1) \) beside \( x = 0 \).

Solve \( mx = x/(x^2 + 1) \) to find \( x^2 + 1 = 1/m \) and then \( x = \pm \sqrt{1/m - 1} \).

So we need that \( 1/m - 1 > 0 \) then

\[
\frac{1}{m} - 1 > 0 \iff \frac{1 - m}{m} > 0 \iff m(m - 1) > 0 \iff 0 < m < 1.
\]

We want to find the area between two curves.

It equals

\[
\int_{\sqrt{1/m-1}}^{\sqrt{1/m-1}} \left| \frac{x}{x^2 + 1} - mx \right| \, dx = 2 \int_{0}^{\sqrt{1/m-1}} \left( \frac{x}{x^2 + 1} - mx \right) \, dx
\]

\[
= 2 \int_{0}^{\sqrt{1/m-1}} \frac{x}{x^2 + 1} \, dx - 2 \int_{0}^{\sqrt{1/m-1}} mxdx = \int_{0}^{\sqrt{1/m-1}} \frac{1}{x^2 + 1} \, dx^2 - 2 \int_{0}^{\sqrt{1/m-1}} mxdx
\]

\[
= (\ln |x^2 + 1| - mx^2)_{\sqrt{1/m-1}} = \ln \frac{1}{m} - m \left( \frac{1}{m} - 1 \right) = m - \ln m - 1.
\]