EX.5

(a) \( f(x) = 1 + x^2 \) and \( \Delta x = \frac{2 - (-1)}{3} = 1 \)

\[ R_3 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) = 1 \cdot 1 + 1 \cdot 2 + 1 \cdot 5 = 8 \]

Next, \( \Delta x = \frac{2 - (-1)}{6} = 0.5 \) \( \Rightarrow \)

\[ R_6 = 0.5 [f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5) + f(2)] \]

\[ = 0.5(1.25 + 1.25 + 2 + 3.25 + 5) = 0.5(13.75) = 6.875 \]

(b) \( L_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(1) = 1.2 + 1.1 + 1.2 = 5 \)

\[ L_6 = 0.5 [f(-1) + f(-0.5) + f(0) + f(0.5) + f(1) + f(1.5)] \]

\[ = 0.5(2 + 1.25 + 1 + 1.25 + 2 + 3.25) = 0.5(10.75) = 5.375 \]

(c) \( M_3 = 1 \cdot f(-0.5) + 1 \cdot f(0) + 1 \cdot f(1.5) \)

\[ = 1 \cdot 1.25 + 1 \cdot 1.25 + 1 \cdot 1.325 = 5.75 \]

\[ M_6 = 0.5 [f(-0.75) + f(-0.25) + f(0.25) + f(0.75) + f(1.25) + f(1.75)] \]

\[ = 0.5(1.5625 + 1.0625 + 1.0625 + 1.5625 + 2.5625 + 4.0625) \]

\[ = 0.5(11.875) = 5.9375 \]

(d) \( M_6 \) appears to be the best estimate.

EX.13

Since \( v \) is an increasing function, \( L_6 \) will give us a lower estimate and \( R_6 \) will give us an upper estimate.

\[ L_6 = (0)(0.5) + (1.9)(0.5) + (3.3)(0.5) + (4.5)(0.5) + (5.5)(0.5) + (5.9)(0.5) \]

\[ = 0.5(21.1) = 10.55 \text{ m}. \]

\[ R_6 = 0.5(1.9 + 3.3 + 4.5 + 5.5 + 5.9 + 6.2) = 0.5(27.3) = 13.65 \text{ m}. \]

EX.23

\[ \lim_{n \to \infty} \sum_{i=1}^{n} \frac{\pi}{4n} \tan \frac{i\pi}{4n} \] can be interpreted as the area of the region lying under the graph of \( y = \tan x \) on the interval \([0, \frac{\pi}{4}]\), since for \( y = \tan x \) on \([0, \frac{\pi}{4}]\) with \( \Delta x = \frac{\pi/4 - 0}{n} = \frac{\pi}{4n} \), \( x_i = 0 + i\Delta x = \frac{i\pi}{4n} \), and \( x_i^* = x_i \), the expression for the area is \( A = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*) \Delta x = \lim_{n \to \infty} \sum_{i=1}^{n} (\tan \frac{i\pi}{4n}) \frac{\pi}{4n} \). Note that this
answer is not unique, since the expression for the area is the same for the function \( y = \tan(x - k\pi) \) on the interval \([k\pi, k\pi + \frac{\pi}{4}]\), where \( k \) is any integer.

EX.25

(a) Since \( f \) is an increasing function, \( L_n \) is an underestimate of \( A \) (lower sum) and \( R_n \) is an overestimate of \( A \) (upper sum). Thus, \( A, L_n, \) and \( R_n \) are related by the inequality \( L_n < A < R_n \).

(b) Let \( a = x_0, x_i = a + i\Delta x \) and \( b = x_n \) where \( \Delta x = \frac{b-a}{n} \).

\[
R_n = f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x
\]

\[
L_n = f(x_0)\Delta x + f(x_1)\Delta x + \cdots + f(x_{n-1})\Delta x
\]

\[
R_n - L_n = f(x_n)\Delta x - f(x_0)\Delta x = \Delta x[f(x_n) - f(x_0)] = \frac{b-a}{n}[f(b) - f(a)].
\]

In the diagram, \( R_n - L_n \) is the sum of the areas of the shaded rectangles. By sliding the shaded rectangles to the left so that they stack on top of the leftmost shaded rectangle, we form a rectangle of height \( f(b) - f(a) \) and width \( \frac{b-a}{n} \).

(c) \( A > l_n, \) so \( R_n - A < R_n - L_n \); that is, \( R_n - A < \frac{b-a}{n}[f(b) - f(a)] \).

EX.30

(a) Consider one of the \( n \) congruent triangles. Let \( O \) be the center of the circle and \( AB \) is one of the sides of the polygon, then \( \triangle AOB \) is one of the \( n \) congruent triangles. Let \( OC \) be the radius which bisects \( \angle AOB \). It follows that \( OC \) intersects \( AB \) at right angles and bisects \( AB \). Thus, \( \triangle AOB \) is divided into two right triangles with legs of length \( \frac{1}{2}(AB) = r\sin\left(\frac{\pi}{n}\right) \) and \( r\cos\left(\frac{\pi}{n}\right) \), where \( r \) is the
radius of the circle. \( \triangle AOB \) has area
\[ 2 \cdot \frac{1}{2} [r \sin(\frac{\pi}{n})][r \cos(\frac{\pi}{n})] = r^2 \sin(\frac{\pi}{n}) \cos(\frac{\pi}{n}) = \frac{1}{2} r^2 \sin(\frac{2\pi}{n}), \]
so \( A_n = n \cdot \text{area}(\triangle AOB) = \frac{1}{2} nr^2 \sin(\frac{2\pi}{n}). \)

(b) To use Equation 3.3.2, \( \lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \), we need to have the same expression in the denominator as we have in the argument of the sine function - in this case, \( \frac{2\pi}{n} \).

\[
\lim_{n \to \infty} A_n = \lim_{n \to \infty} \frac{1}{2} nr^2 \sin(\frac{2\pi}{n}) = \lim_{n \to \infty} \frac{1}{2} nr^2 \sin(s\pi/n) \cdot \frac{2\pi}{n} = \lim_{n \to \infty} \frac{\sin(s\pi/n)}{2\pi/n} \pi r^2.
\]
Let \( \theta = \frac{2\pi}{n} \). Then as \( n \to \infty, \theta \to 0 \),
so \( \lim_{n \to \infty} \frac{\sin(s\pi/n)}{2\pi/n} \pi r^2 = \lim_{\theta \to 0} \frac{\sin \theta}{\theta} \pi r^2 = (1) \pi r^2 = \pi r^2. \)