**EX.1**

Let \( y = f(x) = e^{-x^2} \). First note that \( f(x) \) is an even function, hence symmetric with respect to the \( y \)-axis. Therefore the two vertices of the rectangle we are considering are \( f(x) \) and \( f(-x) \) for some \( x \). The area of the rectangle under the curve from \(-x\) to \( x\) is \( A(x) = 2xe^{-x^2} \) where \( x \geq 0 \).

We maximize \( A(x) \):

\[
A'(x) = 2e^{-x^2} - 4xe^{-x^2} = 2e^{-x^2}(1 - 2x^2) = 0 \Rightarrow x = \frac{1}{\sqrt{2}}.
\]

This gives a maximum since \( A'(x) > 0 \) for \( 0 \leq x < \frac{1}{\sqrt{2}} \) and \( A'(x) < 0 \) for \( x > \frac{1}{\sqrt{2}} \).

We next determine the points of inflection of \( f(x) \). Notice that \( f'(x) = -2xe^{-x^2} = -A(x) \). So \( f''(x) = -A'(x) \) and hence, \( f''(x) < 0 \) for \(-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}} \) and \( f''(x) > 0 \) for \( x < -\frac{1}{\sqrt{2}} \) and \( x > \frac{1}{\sqrt{2}} \).

So \( f(x) \) changes concavity at \( x = \pm \frac{1}{\sqrt{2}} \), and the two vertices of the rectangle of largest area are at the inflection points.

**EX.2**

Let \( f(x) = \sin x - \cos x \) on \([0, 2\pi]\) since \( f \) has period \( 2\pi \).

\[
f'(x) = \cos x + \sin x = 0 \Leftrightarrow \cos x = -\sin x \Leftrightarrow \tan x = -1 \Leftrightarrow x = \frac{3\pi}{4} \text{ or } \frac{7\pi}{4}.
\]

Evaluation \( f \) at its critical numbers and endpoints, we get \( f(0) = -1 \), \( f\left(\frac{3\pi}{4}\right) = \sqrt{2} \), \( f\left(\frac{7\pi}{4}\right) = -\sqrt{2} \), and \( f(2\pi) = -1 \). So \( f \) has absolute maximum value \( \sqrt{2} \) and absolute minimum value \(-\sqrt{2}\). Thus \(-\sqrt{2} \leq \sin x - \cos x \leq \sqrt{2} \Rightarrow |\sin x - \cos x| \leq \sqrt{2} \).

**EX.5**

\[
y = \frac{\sin x}{x} \Rightarrow y' = \frac{x \cos x - \sin x}{x^2} \Rightarrow y'' = \frac{-x^2 \sin x - 2x \cos x + 2 \sin x}{x^3}.
\]

If \((x, y)\) is an inflection point, then \( y'' = 0 \)

\[
\Rightarrow (2 - x^2) \sin x = 2x \cos x \Rightarrow (2 - x^2)^2 \sin^2 x = 4x^2 \cos^2 x
\]

\[
\Rightarrow (2 - x^2)^2 \sin^2 x = 4x^2(1 - \sin^2 x) \Rightarrow (4 - 4x^2 + x^4) \sin^2 x = 4x^2 - 4x^2 \sin^2 x
\]

\[
\Rightarrow (4 + x^4) \sin^2 x = 4x^2 \Rightarrow (x^4 + 4) \sin^2 x = 4
\]

\[
y(x^4 + 4) = 4 \text{ since } y = \frac{\sin x}{x}.
\]
EX.6

\[ y = 1 - x^2 \Rightarrow y' = -2x. \] Let \( P(a, 1 - a^2) \) be the point of contact. The slope of the tangent line at \( P \) is \(-2a\), hence the equation of the tangent line at \( P \) is \( y - (1 - a^2) = (-2a)(x - a) \Rightarrow y - 1 + a^2 = -2a + 2a^2 \Rightarrow y = -2ax + a^2 + 1. \)

To find the \( x \)-intercept, put \( y = 0 \): \( 2ax = a^2 + 1 \Rightarrow x = \frac{a^2 + 1}{2a}. \)

To find the \( y \)-intercept, put \( x = 0 \): \( y = a^2 + 1. \)

Therefore, the area of the triangle is \( \frac{1}{2}(a^2 + 1)(a^2 + 1) = \frac{(a^2 + 1)^2}{4a}. \)

We minimize the function \( A(a) = \frac{(a^2 + 1)^2}{4a}, \ a > 0. \)

\[ A'(a) = \frac{4a(2a^2 + 1)(a^2 + 1) + (a^2 + 1)^2}{16a^2} = \frac{(a^2 + 1)[4a^2 - (a^2 + 1)]}{4a^2} = (a^2 + 1)(3a^2 - 1). \]

\[ A'(a) = 0 \text{ when } 3a^2 - 1 = 0 \Rightarrow a = \frac{1}{\sqrt{3}}, \ A'(a) < 0 \text{ for } a < \frac{1}{\sqrt{3}} \text{ and } A'(a) > 0 \text{ for } a > \frac{1}{\sqrt{3}}. \] So by the First Derivative Test, there is an absolute minimum when \( a = \frac{1}{\sqrt{3}}. \) The required point is \( (\frac{1}{\sqrt{3}}, \frac{2}{3}) \) and the corresponding minimum area is \( A(\frac{1}{\sqrt{3}}) = \frac{4\sqrt{3}}{9}. \)

EX.7

Let \( L = \lim_{x \to 0} \frac{ax^2 + \sin bx + \sin cx + \sin dx}{3x^2 + 5x^4 + 7x^6}. \) Now \( L \) has the indeterminate form of type \( \frac{0}{0} \), so we can apply l’Hospital’s Rule.

\[ L = \lim_{x \to 0} \frac{2ax + b \cos bx + c \cos cx + d \cos dx}{6x + 20x^3 + 42x^5}. \] The denominator approaches 0 as \( x \to 0 \), so the numerator must also approach 0 because the limit exist. Now the numerator approaches \( 0 + b + c + d \), so \( b + c + d = 0. \)

Apply l’Hospital’s Rule again. \( L = \lim_{x \to 0} \frac{2a - b^2 \sin bx - c^2 \sin cx - d^2 \sin dx}{6 + 60x^2 + 210x^4} = \frac{2a - 0}{6 + 0} = \frac{2a}{6}, \) which must equal 8. \( \frac{2a}{6} = 8 \Rightarrow a = 24. \)

Thus, \( a + b + c + d = a + (b + c + d) = 24 + 0 = 24. \)

EX.9

Differentiating \( x^2 + xy + y^2 = 12 \) implicitly with respect to \( x \) gives \( 2x + y + x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0. \) So \( \frac{dy}{dx} = \frac{-2x + y}{x + 2y}. \)

At a highest of lowest point, \( \frac{dy}{dx} = 0 \Leftrightarrow y = -2x. \) Substituting \(-2x\) for \( y \) in the original equation gives \( x^2 + x(-2x) + (-2x)^2 = 12, \) so \( 3x^2 = 12 \) and \( x = \pm 2. \)

If \( x = 2, \) then \( y = -2x = -4, \) and if \( x = -2, \) then \( y = 4. \) Thus, the highest and lowest points are \((-2, 4)\) and \((2, -4). \)
EX.10

\[ y = cx^3 + e^x \Rightarrow y' = 3cx^2 + e^x \Rightarrow y'' = 6cx + e^x. \]

The curve will have inflection points when \( y'' \) changes sign. \( y'' = 0 \Rightarrow -6cx = e^x \), so \( y'' \) will change sign when the line \( y = -6cx \) intersects the curve \( y = e^x \) (but is not tangent to it).

Note that if \( c = 0 \), the curve is just \( y = e^x \), which has no inflection point.

For \( c > 0 \), \( y = -6cx \) will intersect \( y = e^x \) once, so \( y = cx^3 + e^x \) will have one inflection point.

For \( c < 0 \), \( y = -6cx \) can intersect the curve \( y = e^x \) in two points (two inflection points), be tangent to it (no inflection point), or not intersect it (no inflection point). Suppose \( y = -6cx \) is the tangent line at \((a, e^a)\). The tangent line has slope \( e^a \), and it is \( y - e^a = e^a(x - a) \Rightarrow y = e^a(x - ae^a + e^a = -6cx). \) So \(-ae^a + e^a = 0 \Rightarrow a = 1. \) Thus, the slope is \( e \). Next, suppose \( y = -6cx \) intersect \( y = e^x \) in two points, the line \( y = -6cx \) must have slope greater than \( e \), so \(-6c > e \Rightarrow c < -\frac{e}{6} \).

Therefore, the curve \( y = cx^3 + e^x \) will have one inflection point if \( c > 0 \) and two inflection points if \( c < -\frac{e}{6} \).

EX.16

If \( L = \lim_{x \to \infty} \left( \frac{x + a}{x - a} \right)^x \), then \( L \) has the indeterminate form \( 1^\infty \). So

\[
\ln L = \lim_{x \to \infty} \ln \left( \frac{x + a}{x - a} \right)^x = \lim_{x \to \infty} x \ln \left( \frac{x + a}{x - a} \right) = \lim_{x \to \infty} \frac{\ln(x + a) - \ln(x - a)}{1/x}
\]

\[
= \lim_{x \to \infty} \frac{1 + \frac{a}{x} - 1 + \frac{a}{x}}{-1/x^2} \text{ by l’Hospital’s Rule}
\]

\[
= \lim_{x \to \infty} \frac{\frac{2a}{x}}{(x + a)(x - a)} \cdot \frac{-x^2}{1} = \lim_{x \to \infty} \frac{2a}{x^2 - a^2} = \lim_{x \to \infty} \frac{2a}{1 - a^2/x^2} = 2a
\]

Hence, \( \ln L = 2a \), so \( L = e^{2a} \). From the original equation, we want \( L = e^1 \Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2} \).

EX.18

Let the circle have radius \( r \), so \(|OP| = |OQ| = r\), where \( O \) is the center of the circle. Now \( \angle POR \) has measure \( \frac{1}{2} \theta \), and \( \angle OPR \) is a right angle, so \( \tan \frac{1}{2} \theta = \frac{|PR|}{r} \) and the area of \( \triangle OPR \) is \( \frac{1}{2} |OP||PR| = \frac{1}{2} r^2 \tan \frac{1}{2} \theta \). The area of the sector cut by \( OP \) and \( OR \) is \( \frac{1}{2} r^2 \tan \left( \frac{1}{2} \theta \right) = \frac{1}{4} r^2 \theta \). Let \( S \) be the intersection of \( PQ \) and \( OR \). Then \( \sin \frac{1}{2} \theta = \frac{|PS|}{r} \) and \( \cos \frac{1}{2} \theta = \frac{|OS|}{r} \), and the area of \( \triangle OSP \) is \( \frac{1}{2} |OS||PS| = \frac{1}{2} (r \cos \frac{1}{2} \theta)(r \sin \frac{1}{2} \theta) = \frac{1}{2} r^2 \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta = \frac{1}{4} r^2 \sin \theta \).

So \( B(\theta) = 2\left( \frac{1}{4} r^2 \theta - \frac{1}{4} r^2 \sin \theta \right) - r^2 (\tan \frac{1}{2} \theta - \frac{1}{2}) \), and

\( A(\theta) = 2\left( \frac{1}{4} r^2 \theta - \frac{1}{4} r^2 \sin \theta \right) = \frac{1}{2} r^2 (\theta - \sin \theta) \), Thus,
\[
\lim_{\theta \to 0^+} \frac{A(\theta)}{B(\theta)} = \lim_{\theta \to 0^+} \frac{\frac{1}{2} r^2(\theta - \sin \theta)}{r^2(\tan \frac{1}{2} \theta - \frac{1}{2} \theta)} = \lim_{\theta \to 0^+} \frac{\theta - \sin \theta}{2(\tan \frac{1}{2} \theta - \frac{1}{2} \theta)} \\
= \lim_{\theta \to 0^+} \frac{1 - \cos \theta}{2(\frac{1}{2} \sec^2 \frac{1}{2} \theta - \frac{1}{2})} \quad \text{by l'Hospital's Rule} \\
= \lim_{\theta \to 0^+} \frac{1 - \cos \theta}{\sec^2 \frac{1}{2} \theta} = \lim_{\theta \to 0^+} \frac{1 - \cos \theta}{\tan^2 \frac{1}{2} \theta} \\
= \lim_{\theta \to 0^+} \frac{\sin \theta}{2(\tan \frac{1}{2} \theta)(\sec^2 \frac{1}{2} \theta)} \quad \text{by l'Hospital's Rule} \\
= \lim_{\theta \to 0^+} \frac{\sin \theta \cos^3 \frac{1}{2} \theta}{\sin \frac{1}{2} \theta} = \lim_{\theta \to 0^+} \frac{(2 \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta) \cos^3 \frac{1}{2} \theta}{\sin \frac{1}{2} \theta} = 2 \lim_{\theta \to 0^+} \cos^4 \left(\frac{1}{2} \theta\right) = 2(1)^4 = 2.
\]

Remark.

(1) The area of a sector of circle: \( A = \frac{1}{2} r^2 \theta \)

(2) \( \sec^2 \theta - 1 = \tan^2 \theta \)

(3) \( \sec \theta = \frac{1}{\cos \theta} \)

(4) \( \sin 2\theta = 2 \sin \theta \cos \theta \)

**EX.20**

A straight line intersects the curve \( y = f(x) = x^4 + cx^3 + 12x^2 - 5x + 2 \) in four distant points if and only if the graph of \( f \) has two inflection points.

\( f'(x) = 4x^3 + 3cx^2 + 24x - 5 \) and \( f''(x) = 12x^2 + 6cx + 24. \)

\( f''(x) = 0 \iff x = \frac{-6c \pm \sqrt{(6c)^2 - 4(12)(24)}}{2(12)}. \) There are two distinct roots for \( f''(x) = 0 \) (and hence two inflection points) if and only if the discriminant is positive; that is, \( 36c^2 - 1152 > 0 \iff |c| > \sqrt{32}. \)

Thus, the desired values of \( c \) are \( c < -4\sqrt{2} \) or \( c > 4\sqrt{2}. \)