Section 2.5 Continuity

EX.20

Explain why the function is discontinuous at the given number a.

Sketch the graph of the function.

\[ f(x) = \begin{cases} \frac{x^2-x}{x^2-1}, & \text{if } x \neq 1 \\ 1, & \text{if } x = 1 \end{cases} \]

\[ a = 1 \]

\[ \lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = \lim_{x \to 1} \frac{x^2-x}{x^2-1} = \lim_{x \to 1} \frac{x}{x+1} = \frac{1}{2} \neq f(1) = 1 \]

\[ \therefore \text{ } f(x) \text{ is discontinuous at } x = 1 \]

EX.30

Explain, using Theorems 4, 5, 7, and 9, why the function is continuous at every number in its domain.

State the domain.

\[ B(x) = \frac{\tan x}{\sqrt{4-x^2}} \]

\[ \therefore \text{ The domain of } \tan x \text{ is } \mathbb{R} \setminus \{2k\pi \pm \frac{\pi}{2} | k \in \mathbb{Z}\} \text{ and the domain of } \frac{1}{\sqrt{4-x^2}} \text{ is the interval } (-2, 2), \]

\[ \therefore \text{ the domain of } B(x) \text{ is the intersection of the above two domains, namely, } (-2, 2) \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\}. \]

(1) \( \tan x \) is continuous in any interval of the form \( \left( \frac{(2k-1)\pi}{2}, \frac{(2k+1)\pi}{2} \right) \) for all \( k \in \mathbb{Z} \).

(2) \( \frac{1}{2} \) is continuous on \( \mathbb{R} \setminus \{0\} \).

(3) \( \sqrt{x} \) is continuous at every \( x \geq 0 \).

(4) polynomial \( 4-x^2 \) is continuous at every \( x \in \mathbb{R} \).

By (1), (2), (3), and (4), and using Theorems 4, 5, 7, and 9, \( B(x) \) is continuous at every number in its domain \( (-2, 2) \setminus \{-\frac{\pi}{2}, \frac{\pi}{2}\} \).
EX.38

Use continuity to evaluate the limit

$$\lim_{x \to 2} \arctan \left( \frac{x^2 - 4}{3x^2 - 6x} \right)$$

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\lim_{x \to 2} \arctan \left( \frac{x^2 - 4}{3x^2 - 6x} \right) = \arctan \left( \lim_{x \to 2} \frac{(x-2)(x+2)}{3x(x-2)} \right) = \arctan \left( \frac{4}{6} \right) = \arctan \left( \frac{2}{3} \right)$$

EX.42

42. Find the numbers at which \( f \) is discontinuous. At which of these numbers is \( f \) continuous from the right, from the left, or neither? Sketch the graph of \( f \).

\[ f(x) = \begin{cases} 
  x + 1 & \text{if } x \leq 1 \\
  \frac{1}{x} & \text{if } 1 < x < 3 \\
  \sqrt{x - 3} & \text{if } x \geq 3 
\end{cases} \]

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\because (1) \ x + 1 \text{ is continuous everywhere} \\
(2) \frac{1}{x} \text{ is continuous at every } x \neq 0 \\
(3) \sqrt{x - 3} \text{ is continuous at every } x \geq 3 \\
\because \text{we only need to check } x = 1 \land 3 \\
\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} \frac{1}{x} = 1 \neq \lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} (x + 1) = 2 \\
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^+} \frac{1}{x} = \frac{1}{3} \neq \lim_{x \to 3^+} f(x) = \lim_{x \to 3^-} \sqrt{x - 3} = 0 \\
\Rightarrow f(x) \text{ is discontinuous at } x = 1 \land 3

EX.46

Find the values of \( a \) and \( b \) that make \( f \) continuous everywhere.
\[ f(x) = \begin{cases} 
\frac{x^2 - 4}{x - 2}, & \text{if } x < 2 \\
ax^2 - bx + 3, & \text{if } 2 \leq x < 3 \\
2x - a + b, & \text{if } x \geq 3 
\end{cases} \]

It’s easy to check \( f(x) \) continuous at every \( x \neq 2 \& 3 \)

So we only need to check \( x = 2 \& 3 \)

if \( f(x) \) is continuous at everywhere

then \( \lim_{x \to 2} f(x) = f(2) \& \lim_{x \to 3} f(x) = f(3) \)

\[
\begin{align*}
\lim_{x \to 2} f(x) &= \begin{cases} 
\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} ax^2 - bx + 3 = 4a - 2b + 3 \\
\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} \frac{(x-2)(x+2)}{x-2} = 4
\end{cases} \\
\lim_{x \to 3} f(x) &= \begin{cases} 
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} 2x - a + b = 6 - a + b \\
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} ax^2 - bx + 3 = 9a - 3b + 3
\end{cases}
\end{align*}
\]

\( \Rightarrow \begin{cases} 4a - 2b + 3 = 4 \\
9a - 3b + 3 = 6 - a + b \end{cases} \Rightarrow \begin{cases} a = \frac{1}{2} \\
b = \frac{1}{2} \end{cases} \)

**EX.54**

Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

\[ \sin x = x^2 - x, \quad (1, 2) \]

\( \sin x \) and \( x^2 - x \) are both continuous at \( (1, 2) \)

Define \( f(x) = \sin x - (x^2 - x) \)

then \( f(1) = \sin 1 - (1 - 1) = \sin 1 > 0 \)

\( f(2) = \sin 2 - (4 - 2) = \sin 2 - 2 < 0 \)

\( \therefore f(1)f(2) < 0 \)

\( \therefore \) By I.V.T there exist a \( c \in (1, 2) \) s.t \( f(c) = 0 \)
EX.64

For what values of $x$ is $g$ continuous

$$g(x) = \begin{cases} 
0 & \text{if } x \text{ is rational} \\
x & \text{if } x \text{ is irrational}
\end{cases}.$$

Claim: $g(x)$ is continuous only at $x = 0$.

Given $\varepsilon > 0$, take $\delta = \varepsilon > 0$.

If $0 < |x - 0| < \delta$,

then

$$|g(x) - g(0)| = \begin{cases} 
|0 - 0| = 0 < \varepsilon & \text{if } x \text{ is rational} \\
|x - 0| < \delta = \varepsilon & \text{if } x \text{ is irrational}
\end{cases}.$$

$\therefore \lim_{x \to 0} g(x) = g(0)$.

For any $c \neq 0$

(1) If $c$ is irrational,

take $\varepsilon = \frac{|c|}{2} > 0$. For all $\delta > 0$, there exists a rational number $x$ such that $0 < |x - c| < \delta$
(because the rational numbers are dense in the real numbers) and

$$|g(x) - g(c)| = |0 - c| = |c| > \varepsilon.$$

(2) If $c$ is rational and positive,

take $\varepsilon = |c| > 0$. For all $\delta > 0$, there exists an irrational number $x \in (c, c + \delta)$ (because the irrational numbers are dense in the real numbers) and

$$|g(x) - g(c)| = |x - 0| = |x| > |c| = \varepsilon.$$

(3) If $c$ is rational and negative,

take $\varepsilon = |c| > 0$. For all $\delta > 0$, there exists an irrational number $x \in (c - \delta, c)$ (because the irrational numbers are dense in the real numbers) and

$$|g(x) - g(c)| = |x - 0| = |x| > |c| = \varepsilon.$$

By (1), (2), and (3), $\therefore g(x)$ is discontinuous at every $x \neq 0$. 
EX.65

Is there a number that is exactly 1 more than its cube?

<pf>
Yes,
Define \( f(x) = x - (1 + x^3) \)
\( f(-1) = -1 \land f(-2) = 5 \)
\( \therefore f(x) \) is continuous at everywhere
\( \therefore \) By I.V.T \( \forall t \in (-1, 5) \exists c \in (-2, -1) \) s.t \( f(c) = t \)
i.e \( \exists c \in (-2, -1) \) s.t \( f(c) = 0 \)
\( \Rightarrow c - (1 + c^3) = 0 \Rightarrow c = 1 + c^3 \)

EX.66

If \( a \) and \( b \) are positive numbers, prove that the equation

\[
\frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} = 0
\]

has at least one solution in the interval \((-1, 1)\).

<pf>
\[ x^3 + 2x^2 - 1 = (x + 1)(x^2 + x - 1) \land x^3 + x - 2 = (x - 1)(x^2 + x + 2) \]
Define \( f(x) = \frac{a}{x^3 + 2x^2 - 1} + \frac{b}{x^3 + x - 2} \)
\[
\lim_{x \to 1^-} f(x) = \frac{a}{2} + \lim_{x \to 1^-} \frac{b}{x^3 + x - 2} \Rightarrow \lim_{x \to 1^-} \frac{b}{x^3 + x - 2} = -\infty
\]
\[
\lim_{x \to (\sqrt{5} - 1)^+} f(x) = \lim_{x \to (\sqrt{5} - 1)^+} \frac{a}{x^3 + 2x^2 - 1} + \frac{b}{3\sqrt{5} - 2} \Rightarrow \lim_{x \to (\sqrt{5} - 1)^+} \frac{a}{x^3 + 2x^2 - 1} = \infty
\]
Given \( \varepsilon > 0 \) \( \exists x_1, x_2 \in (-1, 1) \)
s.t \( 1 - x_1 < \varepsilon \land x_2 - \frac{\sqrt{5} - 1}{2} < \varepsilon \)
satisfy \( f(x_1) < 0 \land f(x_2) > 0 \)
\( \Rightarrow f(x) \) has at least one solution in the interval \((-1, 1)\).