Solution of Exercise 16.3

EX.14

\[ \mathbf{F} = (1 + xy)e^{xy}\hat{i} + x^2e^{xy}\hat{j}. \]
(a) Since \((1+xy)e^{xy}\) is continuous w.r.t. \(x\) on \(\mathbb{R}\), \(\int (1+xy)e^{xy}dx = \frac{1}{y}e^{xy} + \frac{1}{y}e^{xy}xy - \int e^{xy}dx. \)
Let \(f(x, y) = xe^{xy} + C(y)\) with \(\frac{\partial f}{\partial y} = x^2e^{xy}\), then \(x^2e^{xy} + C'(y) = x^2e^{xy} \Rightarrow C'(y) = 0\)
Let \(f(x, y) = xe^{xy}(i.e.C(y) = 0)\), then clearly \(f\) is differentiable on \(\mathbb{R}^2\) and \(\nabla f = \mathbf{F}\).
(b) By the thm, \(\int_C \mathbf{F} \cdot d\mathbf{r} = f(0, 2) - f(1, 0) = -1.\)

EX.15

(a) Since \(yz\) is continuous w.r.t. \(x\) on \(\mathbb{R}\), \(\int yzdx = xyz + C(x, y)\)
Let \(f(x, y, z) = xyz + C(x, y)\) with \(\frac{\partial f}{\partial y} = xz, \frac{\partial f}{\partial z} = xy + 2z\), then \(\frac{\partial f}{\partial y} = xz + \frac{\partial C}{\partial y} = xz \Rightarrow \frac{\partial C}{\partial y} = 0 \Rightarrow C = C(z); \frac{\partial f}{\partial z} = xy + \frac{\partial C}{\partial z} = xy + 2z \Rightarrow \frac{\partial C}{\partial z} = 2z \Rightarrow C(x, y) = z^2 + C.\)
Let \(f(x, y) = xyz + z^2(i.e.C = 0)\), then clearly \(f\) is differentiable on \(\mathbb{R}^3\) and \(\nabla f = \mathbf{F}\).
(b) By the thm, \(\int_C \mathbf{F} \cdot d\mathbf{r} = f(4, 6, 3) - f(1, 0, -2) = 81 - 4 = 77.\)

EX.16

(a) Since \(y^2z + 2xz^2\) is continuous w.r.t. \(x\) on \(\mathbb{R}\), \(\int y^2z + 2xz^2dx = xy^2z + x^2z^2 + C(y, z)\).
Let \(f(x, y, z) = xy^2z + x^2z^2C(y, z)\), with \(\frac{\partial f}{\partial y} = 2xyz; \frac{\partial f}{\partial z} = xy2 + 2x^2z\), then \(\frac{\partial f}{\partial y} = 2xyz + \frac{\partial C}{\partial y} = 2xyz \Rightarrow \frac{\partial C}{\partial y} = 0 \Rightarrow C(y, z) = C(z). \frac{\partial f}{\partial z} = xy^2 + 2x^2z + \frac{\partial C}{\partial z} = xy^2 + 2x^2z \Rightarrow \frac{\partial C}{\partial z} = 0 \Rightarrow C(z) = C.\)
Let \(f(x, y, z) = xy^2z + x^2z^2(i.e.C = 0)\), then clearly \(f\) is differentiable on \(\mathbb{R}\) and \(\nabla f = \mathbf{F}\).
(b) By the thm, \(\int_C \mathbf{F} \cdot d\mathbf{r} = f(1, 2, 1) - f(0, 1, 0) = 4 + 1 = 5.\)
EX.19

Let $\vec{F}(x, y) = \tan \hat{t}i + x \sec^2 y \hat{j}$, then since $\vec{r} = \hat{x}i + y \hat{j} \Rightarrow d\vec{r} = \hat{i}dx + \hat{j}dy \Rightarrow \tan ydy + x \sec^2 dy = \vec{F} \cdot d\vec{r}$.

Since $\tan y$ is cont. w.r.t. $x$ on $\mathbb{R}$, $\int \tan ydy = x \tan y + C(y)$. Let $f(x, y) = x \tan y + C(y)$ with $\frac{\partial f}{\partial y} = x \sec^2 y$, then $\frac{\partial f}{\partial y} = x \sec^2 y + \frac{\partial C}{\partial y} = x \sec^2 y \Rightarrow \frac{\partial C}{\partial y} = 0 \Rightarrow C(y) = C$. Let $f(x, y) = x \tan y (i.e. C(y) = 0)$, then clearly, $f$ is diff. on $\mathbb{R}^2$ and $\nabla f = \vec{F}$. By the thm, $\int_C \vec{F} \cdot d\vec{r} = f(2, \frac{\pi}{4}) - f(1, 0) = 2 \Rightarrow$ the integral is independ. of path.

EX.20

Let $\vec{F}(x, y) = (1 - ye^{-x})\hat{i} + e^{-x}\hat{j}$, then since $\vec{r} = \hat{x}i + y \hat{j} \Rightarrow d\vec{r} = \hat{i}dx + j \hat{y}dy$, $(1 - ye^{-x})dx + e^{-x}dy = \vec{F} \cdot d\vec{r}$.

Since $1 - ye^{-x}$ is conti. w.r.t. $x$ on $\mathbb{R}$, $\int 1 - ye^{-x}dx = x + ye^{-x} + C(y)$. Let $f(x, y) = x + ye^{-x} + C(y)$, with $\frac{\partial f}{\partial y} = e^{-x}$, then $\frac{\partial f}{\partial y} = e^{-x} + \frac{\partial C}{\partial y} = e^{-x} \Rightarrow \frac{\partial C}{\partial y} = 0 \Rightarrow C(y) = C$.

Let $f(x, y) = f(x, y) = x + ye^{-x}$, then clearly, $f$ is diff. on $\mathbb{R}^2$ and $\nabla f = \vec{F}$. By the thm, $\int_C \vec{F} \cdot d\vec{r} = f(1, 2) - f(0, 1) = (1 + e-1) - 1 = \frac{2}{e}$. The integral is independ. of path.

EX.23

claim: $\vec{F}$ is conservative on $\mathbb{R}^2$.

Since $2y^2 \frac{\hat{i}}{2}$ is conti. w.r.t. $x$ on $\mathbb{R}$, $\int 2y^2 \frac{\hat{i}}{2}dx = 2xy \hat{j} + C(y)$. Let $f(x, y) = 2xy \hat{j} + C(y)$, with $\frac{\partial f}{\partial y} = 3x \sqrt{y}$, then $\frac{\partial f}{\partial y} = 3x \sqrt{y} + \frac{\partial C}{\partial y} = 3x \sqrt{y} \Rightarrow C(y) = C$.

Let $f(x, y) = 2xy \hat{j}$, then $f$ is diff. on $\mathbb{R}$ and $\nabla f = \vec{F}$. If $C$ is the path of the object, then by the thm, $w = \int_C \vec{F} \cdot d\vec{r} = f(2, 4) - f(1, 1) = 32 - 2 = 30$.

EX.24

claim: $\vec{F}$ is conservative on $\mathbb{R}^2$.

Since $e^{-y}$ is conti. w.r.t. $x$ on $\mathbb{R}$, $\int e^{-y} dx = xe^{-y} + C(y)$. Let $f(x, y) = xe^{-y} + C(y)$, with $\frac{\partial f}{\partial y} = xe^{-y}$, then $\frac{\partial f}{\partial y} = xe^{-y} + \frac{\partial C}{\partial y} = xe^{-y} \Rightarrow C(y) = C$.

Let $f(x, y) = xe^{-y}$, then $f$ is diff. on $\mathbb{R}^2$ and $\nabla f = \vec{F}$. If $C$ is the path of the object, then by the thm, $w = \int_C \vec{F} \cdot d\vec{r} = f(2, 0) - f(0, 1) = 2$. 

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EX.35

(a) $P = -\frac{y}{x^2+y^2}$; $Q = \frac{x}{x^2+y^2}$.

Since $-y$ and $x^2+y^2$ are diff. w.r.t. $x$ on $\mathbb{R}$, we have $\frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2}$. Similarly, $x$ and $x^2+y^2$ are diff. w.r.t. $y$ on $\mathbb{R}$, we have $\frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$.

Thus, $\frac{\partial P}{\partial x} = \frac{\partial Q}{\partial y}$.

(b) On the upper half circle, $y = \sqrt{1-x^2} \Rightarrow dy = \frac{-x}{\sqrt{1-x^2}}dx$

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} \left( \frac{x}{\sqrt{1-x^2}} \right) dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx.
\]

Let $x = \sin z$, then the integral becomes $\int_{\frac{\pi}{2}}^{\frac{-\pi}{2}} \frac{-1}{\cos z} (\cos z) dz = -\pi$

On the lower half circle, $y = -\sqrt{1-x^2} \Rightarrow dy = \frac{x}{\sqrt{x^2}} dx$,

\[
\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_{-1}^{1} \frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} \left( \frac{x}{\sqrt{1-x^2}} \right) dx = \int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} dx = \pi.
\]

No, it does not. By the thm, every simple-connected region which contains $C_1$ and $C_2$ would avoid the origin. But, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} = \frac{y^2-x^2}{(x^2+y^2)^2}$ is not conti. at $(0,0) \Rightarrow P$ and $Q$ are not have conti. first-order derivatives.