Section 14.6 Directional Derivatives and the Gradient Vector

EX.5

\[ f(x, y) = ye^{-x} \Rightarrow f_x(x, y) = -ye^{-x} \text{ and } f_y(x, y) = e^{-x}. \]

If \( u \) is a unit vector in the direction of \( \theta = 2\pi/3 \), then from Equation 6, \( D_u f(0, 4) = f_x(0, 4) \cos \left( \frac{2\pi}{3} \right) + f_y(0, 4) \sin \left( \frac{2\pi}{3} \right) = -4 \cdot \left( -\frac{1}{2} \right) + 1 \cdot \frac{\sqrt{3}}{2} = 2 + \frac{\sqrt{3}}{2}. \)

EX.9

\[ f(x, y, z) = xe^{2yz} \]

(a) \( \nabla f(x, y, z) = \langle f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) \rangle = \langle e^{2yz}, 2xe^{2yz}, 2xye^{2yz} \rangle \)

(b) \( \nabla f(3, 0, 2) = \langle 12, 12, 0 \rangle \)

(c) By Equation 14, \( D_u f(3, 0, 2) = \nabla f(3, 0, 2) \cdot u = 12 \cdot \left( \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right) = \frac{22}{3}. \)

EX.15

\[ f(x, y, z) = xe^{y} + ye^{z} + ze^{x} \Rightarrow \nabla f(x, y, z) = \langle e^{y} + ze^{z}, xe^{y} + e^{z}, ye^{z} + e^{x} \rangle, \nabla f(0, 0, 0) = \langle 1, 1, 1 \rangle, \]

and a unit vector in the direction of \( v \) is \( u = \frac{1}{\sqrt{25+1+4}} \langle 5, 1, -2 \rangle = \frac{1}{\sqrt{30}} \langle 5, 1, -2 \rangle \), so \( D_u f(0, 0, 0) = \nabla f(0, 0, 0) \cdot u = \langle 1, 1, 1 \rangle \cdot \frac{1}{\sqrt{30}} \langle 5, 1, -2 \rangle = \frac{4}{\sqrt{30}}. \)

EX.23

\[ f(x, y) = \sin(xy) \Rightarrow \nabla f(x, y) = \langle y \cos(xy), x \cos(xy) \rangle, \nabla f(1, 0) = \langle 0, 1 \rangle. \] Thus the maximum rate of change is \( |\nabla f(1, 0)| = 1 \) in the direction \( \langle 0, 1 \rangle. \)

EX.39

\[ f(x, y) = x^3 + 5x^2y + y^3 \Rightarrow \]

\[ D_u f(x, y) = \nabla f(x, y) \cdot u = \langle 3x^2 + 10xy, 5x^2 + 3y^2 \rangle \cdot \left( \frac{3}{5}, \frac{4}{5} \right) \]

\[ = \frac{9}{5}x^2 + 6xy + 4x^2 + \frac{12}{5}y^2 = \frac{29}{5}x^2 + 6xy + \frac{12}{5}y^2. \] Then

\[ D_u^2 f(x, y) = D_u [D_u f(x, y)] = \nabla [D_u f(x, y)] \cdot u \]

\[ = \langle \frac{58}{25}x + 6y, \frac{24}{5}y \rangle \cdot \left( \frac{3}{5}, \frac{4}{5} \right) = \frac{294}{25}x + \frac{186}{25}y \]

and \( D_u^2 f(2, 1) = \frac{294}{25}(2) + \frac{186}{25}(1) = \frac{774}{25}. \)
EX.45

Let $F(x, y, z) = x + y + z - e^{xyz}$. Then

$x + y + z = e^{xyz}$ is the level surface $F(x, y, z) = 0$, and

$\nabla F(x, y, z) = (1 - yze^{xyz}, 1 - xze^{xyz}, 1 - yze^{xyz})$.

(a) $\nabla F(0, 0, 1) = (1, 1, 1)$ is a normal vector for the tangent plane at $(0, 0, 1)$, so an equation of the tangent plane is

$$1(x - 0) + 1(y - 0) + 1(z - 1) = 0 \text{ or } x + y + z = 1.$$

(b) The normal line has direction $(1, 1, 1)$, so parametric equations are $x = t, y = t, z = 1 + t$, and symmetric equations are $x = y = z - 1$.

EX.51

$\nabla F(x_0, y_0, z_0) = \langle \frac{2x_0}{a^2}, \frac{2y_0}{b^2}, \frac{2z_0}{c^2} \rangle$. Thus an equation of the tangent plane at $(x_0, y_0, z_0)$ is

$$\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z = 2 \left( \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} \right) = 2(1) = 2 \text{ since } (x_0, y_0, z_0) \text{ is a point on the ellipsoid. Hence }$$

$$\frac{2x_0}{a^2}x + \frac{2y_0}{b^2}y + \frac{2z_0}{c^2}z = 1$$

is an equation of the tangent plane.

EX.56

First note that the point $(1,1,2)$ is on both surfaces. The ellipsoid is a level surface of $F(x, y, z) = 3x^2 + 2y^2 + z^2$ and $\nabla F(x, y, z) = \langle 6x, 4y, 2z \rangle$. A normal vector to the surface at $(1, 1, 2)$ is $\nabla F(1, 1, 2) = \langle 6, 4, 4 \rangle$ and an equation of the tangent plane there is $6(x - 1) + 4(y - 1) + 4(z - 2) = 0$ or $6x + 4y + 4z = 18$ or $3x + 2y + 2z = 9$. The sphere is a level surface of $G(x, y, z) = x^2 + y^2 + z^2 - 8x - 6y - 8z + 24$ and $\nabla G(x, y, z) = \langle 2x - 8, 2y - 6, 2z - 8 \rangle$. A normal vector to the sphere at $(1, 1, 2)$ is $\nabla G(1, 1, 2) = \langle -6, -4, -4 \rangle$ and the tangent plane there is $-6(x - 1) - 4(y - 1) - 4(z - 2) = 0$ or $3x + 2y + 2z = 9$.

Since these tangent planes are identical, the surfaces are tangent to each other at the point $(1, 1, 2)$.

EX.61

Let $(x_0, y_0, z_0)$ be a point on the surface. Then an equation of the tangent plane at the point is

$$\frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \frac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{\sqrt{2 \sqrt{2}}} \cdot$$

But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is

$$\frac{x}{\sqrt{x_0}} + \frac{y}{\sqrt{y_0}} + \frac{z}{\sqrt{z_0}} = \sqrt{c} \cdot$$

The $x$-, $y$-, and $z$-intercepts are $\sqrt{c x_0}$, $\sqrt{c y_0}$, and $\sqrt{c z_0}$ respectively. (The $x$-intercept is found by setting $y = z = 0$ and solving the resulting equation for $x$, and the $y$- and $z$-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c \left( \sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} \right)} = c$, a constant.
EX.65

(a) The direction of the normal line of $F$ is given by $\nabla F$, and that of $G$ by $\nabla G$. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at $P$ if $\nabla F \cdot \nabla G = 0$ at $P$

\[ \Rightarrow \langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z \rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z \rangle = 0 \text{ at } P \]

\[ \Rightarrow F_x G_x + F_y G_y + F_z G_z = 0 \text{ at } P. \]

(b) Here $F = x^2 + y^2 - z^2$ and $G = x^2 + y^2 + z^2 - r^2$, so

$\nabla F \cdot \nabla G = \langle 2x, 2y, -2z \rangle \cdot \langle 2x, 2y, 2z \rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point $(x, y, z)$ lies on the graph of $F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere’s normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone’s normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.