Chapter 13 Problems Plus

EX.1

(a) For a fixed \( v_0 \) and a given \( \alpha \), the height

\[
h(t) = y(t) = v_0 \sin \alpha t - \frac{1}{2} gt^2
\]

\[
h'(t) = v_0 \sin \alpha - gt = 0 \text{ when } t = \frac{v_0 \sin \alpha}{g} \triangleq t_0
\]

The corresponding maximum height is \( h(t_0) = \frac{v_0^2 \sin^2 \alpha}{2g} \), which attains its maximum value of \( \frac{v_0^2}{2g} \) when \( \alpha = \frac{\pi}{2} \).

(b)

\[
\begin{align*}
x(t) &= v_0 (\cos \alpha) t \Rightarrow t = \frac{x}{v_0 \cos \alpha} \\
y(t) &= v_0 \sin \alpha t - \frac{1}{2} gt^2
\end{align*}
\]

\[\Rightarrow y = x \tan \alpha - \frac{g}{2v_0^2 \cos^2 \alpha} x^2\]

Given \( v_0 \), the maximal (horizontal) range has been shown to be \( R = \frac{v_0^2}{g} \).

For a fixed \( v_0 \) and given \( x_0 \in [0, R] \) (all possible values), we want to find the maximum achievable height right above \( x = x_0 \) as follows:

At \( x = x_0 \), the height is

\[
y(\alpha) = x_0 \tan \alpha - \frac{x_0^2}{2R \cos^2 \alpha} = x_0 \tan \alpha - \frac{x_0^2 \sec^2 \alpha}{2R}
\]

And

\[
y'(\alpha) = x_0 \sec^2 \alpha - \frac{x_0^2}{2R} \cdot 2 \sec^2 \alpha \tan \alpha
\]

\[y'(\alpha) = 0 \text{ when } \alpha = \tan^{-1}(R/x_0) \triangleq \alpha_0. \text{ In this case}
\]

\[y_{\text{max}} = y(\alpha_0) = \frac{R}{2} - \frac{x_0}{2R}
\]

\[\Rightarrow x_0^2 + 2Ry - R^2 = 0
\]

which is on the given boundary. Other nonnegative heights smaller than this value are also achievable by adjusting \( \alpha \).
When the projectile is right above \( x = D \)

\[
x(t_0) = v_0(\cos \alpha)t_0 \triangleq D
\]

\[
\Rightarrow t_0 = \frac{D}{v_0 \cos \alpha}
\]

At this time, the height of the projectile is

\[
y_1(t_0) = v_0 \sin \alpha \left( \frac{D}{v_0 \cos \alpha} \right) - \frac{1}{2}g \left( \frac{D}{v_0 \cos \alpha} \right)^2
\]

\[
= D \tan \alpha - \frac{gD^2}{2v_0^2 \cos^2 \alpha}
\]

\[
= h - \frac{gD^2}{2v_0^2 \cos^2 \alpha}
\]

since \( \tan \alpha = D/h \) by the figure above.

On the other hand, the height of the target is

\[
y_2(t_0) = h - \frac{1}{2}gt^2 = y_1(t_0)
\]

Thus the projectile always hit the target, provided the projectile does not hit the ground before \( D \).

**EX.4**

\[
r(t) = \left\langle \int_0^t \sin \left( \frac{1}{2} \pi \theta^2 \right) d\theta, \int_0^t \cos \left( \frac{1}{2} \pi \theta^2 \right) d\theta \right\rangle
\]

By the Fundamental Theorem of Calculus,

\[
r'(t) = \left\langle \sin \left( \frac{1}{2} \pi t^2 \right), \cos \left( \frac{1}{2} \pi t^2 \right) \right\rangle
\]

\[
= T(t) \text{ since } |r'(t)| = 1
\]

\[
T'(t) = \left\langle \cos \left( \frac{1}{2} \pi t^2 \right) \cdot \frac{1}{2} \pi \cdot 2t, -\sin \left( \frac{1}{2} \pi t^2 \right) \cdot \frac{1}{2} \pi \cdot 2t \right\rangle
\]

\[
= \left\langle \pi t \cos \left( \frac{1}{2} \pi t^2 \right), -\pi t \sin \left( \frac{1}{2} \pi t^2 \right) \right\rangle
\]

\[
|T'(t)| = \pi |t|
\]

\[
\therefore \kappa = \frac{|T'(t)|}{|r'(t)|} = \pi |t|
\]
EX.6

As the cable wrap around the spool, its axis forms a helix of radius \((R + r)\). Let \(h\) be the vertical distance between coils (see the figure), from similar triangles we have

\[
\frac{2r}{\sqrt{h^2 - 4r^2}} = \frac{2\pi(R + r)}{h}
\]

\[\Rightarrow h = \frac{2\pi r(R + r)}{\sqrt{\pi^2(R + r)^2 - r^2}}\]

The helix can be parametrized by

\[
r(\theta) = \left< (R + r) \cos \theta, (R + r) \sin \theta, \frac{\theta}{2\pi} h \right>
\]

\[
r(\theta) = \left< -(R + r) \sin \theta, (R + r) \cos \theta, \frac{h}{2\pi} \right>
\]

With the above value of \(h\), the length of a complete cycle of cable is

\[
l = \int_0^{2\pi} |r'(\theta)| d\theta
\]

\[
= \int_0^{2\pi} \sqrt{(R + r)^2 + \left(\frac{h}{2\pi}\right)^2} d\theta
\]

\[
= 2\pi \sqrt{(R + r)^2 + \left(\frac{h}{2\pi}\right)^2}
\]

\[
= \frac{2\pi^2(R + r)^2}{\sqrt{\pi^2(R + r)^2 - r^2}}
\]

The number of complete cycles is \(\lfloor L/l \rfloor\), and thus the shortest length along the spool is

\[
h \cdot \left| \frac{L}{l} \right| = \frac{2\pi r(R + r)}{\sqrt{\pi^2(R + r)^2 - r^2}} \left[ \frac{L\sqrt{\pi^2(R + r)^2 - r^2}}{2\pi^2(R + r)^2} \right]
\]