EX.7

Let \( f(x) = \frac{1}{x\sqrt{\ln x}} \). \( f \) is positive, continuous and decreasing on \([2, \infty)\). \( \int_2^\infty \frac{1}{x\sqrt{\ln x}} \, dx = [2\sqrt{\ln x}]_2^\infty = \infty \), so the given series is divergent by the Integral Test.

EX.17

Let \( a_n = \frac{n!}{2 \cdot 5 \cdot 8 \cdots (3n + 2)} \). Apply the Ratio Test,

\[
\lim_{n \to \infty} \frac{|a_{n+1}|}{a_n} = \lim_{n \to \infty} \frac{(n+1)!}{2 \cdot 5 \cdot 8 \cdots (3n + 5)} \frac{2 \cdot 5 \cdot 8 \cdots (3n + 2)}{n!} = \lim_{n \to \infty} \frac{n+1}{3n + 5} = \frac{1}{3} < 1
\]

Therefore the given series is (absolutely) convergent by the Ratio Test.

EX.24

Because \( \lim_{n \to \infty} n \sin \frac{1}{n} = \lim_{x \to 0} \frac{\sin x}{x} = 1 \neq 0 \), the given series diverges.

EX.32

Let \( a_n = \frac{(n!)^n}{n^{4n}} \). Apply the Root Test,

\[
\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \frac{n!}{n^4} = \lim_{n \to \infty} \frac{(n-4)!}{n^4/n(n-1)(n-2)(n-3)} = \infty
\]

Therefore the given series is divergent by the Root Test.

EX.36

\((\ln n)^{\ln n} = (e^{\ln n})^{\ln n} = (e^{\ln n})^{\ln \ln n} = n^{\ln \ln n}\). Because

\(\lim_{n \to \infty} \ln \ln n = \infty \Rightarrow \exists \) a integer \( N \) such that \( \ln \ln n > 2 \) for all \( n > N \)

we have \( \frac{1}{(\ln n)^{\ln n}} = \frac{1}{n^{\ln \ln n}} < \frac{1}{n^2} \) for all \( n > N \). Since \( \sum_{n=2}^\infty \frac{1}{n^2} \) is a convergent \( p \)-series \((p = 2 > 1)\), the given series is (absolutely) convergent by the Comparison Test.
EX.37

Let $a_n = (\sqrt[2]{2} - 1)^n$. Apply the Root Test,

$$\lim_{n \to \infty} \sqrt[n]{|a_n|} = \lim_{n \to \infty} \sqrt[2]{2} - 1 = 1 - 1 = 0 < 1$$

Therefore the given series is (absolutely) convergent by the Root Test.