Section 11.3 The Integral Test and Estimates of Sums

**EX.2**

From the two figure, we can see that \( \int_1^6 f(x)dx < \sum_{i=1}^{5} a_i \) & \( \int_1^6 f(x)dx > \sum_{i=2}^{6} a_i \)

Thus, \( \sum_{i=2}^{6} a_i < \int_1^6 f(x)dx < \sum_{i=1}^{5} a_i \).

**EX.6**

\( f(x) = \frac{1}{\sqrt{x+4}} = (x+4)^{-1/2} \) is continuous, positive, and decreasing on \([1, \infty)\)

\[
\int_1^\infty (x+4)^{-1/2}dx = \lim_{t \to \infty} \int_1^t (x+4)^{-1/2}dx = \lim_{t \to \infty} \left[ 2(x+4)^{1/2} \right]_1^t = \lim_{t \to \infty} \left[ 2\sqrt{t+4} - 2\sqrt{5} \right] = \infty
\]

So, \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+4}} \) is divergent by the integral test.

**EX.18**

\( f(x) = \frac{3x-4}{x^2-2x} = \frac{2}{x} + \frac{1}{x-2} \) is continuous, positive, and decreasing on \([3, \infty)\)

\[
\int_3^\infty \frac{3x-4}{x^2-2x}dx = \lim_{t \to \infty} \int_3^t \left[ \frac{2}{x} + \frac{1}{x-2} \right]dx = \lim_{t \to \infty} \left[ 2\ln x + \ln(x-2) \right]_3^t = \lim_{t \to \infty} \left[ 2\ln t + \ln(t-2) - 2\ln 3 \right] = \infty
\]

So, \( \sum_{n=3}^{\infty} \frac{3n-4}{n^2-2n} \) is divergent by the integral test.

**EX.21**

\( f(x) = \frac{1}{x \ln x} \) is continuous, positive, and decreasing on \([2, \infty)\)

\( (f'(x) = -\frac{1+\ln x}{x^2(\ln x)^2} < 0 \) for \( x > 2 \))

\[
\int_2^\infty \frac{1}{x \ln x}dx = \lim_{t \to \infty} \left[ \ln(\ln x) \right]_2^t = \lim_{t \to \infty} \left[ \ln(\ln t) - \ln(\ln 2) \right] = \infty
\]

So, \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) is divergent by the integral test.
EX.22

\[ f(x) = \frac{1}{x^2 \ln x} \] is continuous, positive, and decreasing on \([2, \infty)\)

\[
\int_{2}^{\infty} \frac{1}{x^2 \ln x} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{\ln x} \right]_{2}^{t} = - \lim_{t \to \infty} \left[ \ln t - \ln 2 \right] = \frac{1}{\ln 2}
\]

So, \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) is convergent by the integral test.

EX.29

By EX.21, we know that when \( p = 1 \) the series \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) is divergent. So we may assume that \( p \neq 1 \).

\[ f(x) = \frac{1}{x^{p} \ln x} \] is continuous, positive on \([2, \infty)\), and \( f'(x) = -\frac{p+\ln x}{x^{2}(\ln x)^{p+1}} < 0 \) if \( x > e^{-p} \), so that \( f \) is eventually decreasing.

\[
\int_{2}^{\infty} \frac{1}{x^{p} \ln x} \, dx = \lim_{t \to \infty} \left[ \frac{(\ln x)^{1-p}}{1-p} \right]_{2}^{t} = \lim_{t \to \infty} \left[ \frac{(\ln t)^{1-p}}{1-p} - \frac{(\ln 2)^{1-p}}{1-p} \right]
\]

which exists whenever \( 1 - p < 0 \), so the series is convergent for \( p > 1 \).

EX.30

\[ f(x) = \frac{1}{x \ln x [\ln(\ln x)]^{p}} \] is continuous, positive on \([3, \infty)\). For \( p \geq 0 \), it is clearly that \( f \) is decreasing on \([3, \infty)\); and for \( p < 0 \), we can verify that \( f'(x) < 0 \) whenever \( x \) is greater than some \( l \in \mathbb{R} \). So we apply the integral test now.

\[
I = \int_{3}^{\infty} \frac{1}{x \ln x [\ln(\ln x)]^{p}} \, dx = \lim_{t \to \infty} \int_{3}^{t} \frac{[\ln(\ln x)]^{-p}}{x \ln x} \, dx = \lim_{t \to \infty} \left[ \frac{[\ln(\ln x)]^{-p+1}}{-p+1} \right]_{3}^{t}
\]

(for \( p \neq 1 \))

\[
= \lim_{t \to \infty} \left[ \frac{[\ln(\ln t)]^{-p+1}}{-p+1} - \frac{[\ln(\ln 3)]^{-p+1}}{-p+1} \right], \text{ which exists whenever } -p + 1 > 0 \Leftrightarrow p > 1.
\]

If \( p = 1 \), then \( I = \lim_{t \to \infty} [\ln(\ln x)]_{3}^{t} = \infty \).

So, the series is convergent for \( p > 1 \).
EX.32

If \( p \leq 0 \), \( \lim_{n \to \infty} \frac{\ln n}{n^p} = \infty \), then the series is divergent.

So, we may assume that \( p > 0 \), \( f(x) = \frac{\ln x}{x^p} \) is continuous, positive on \([2, \infty)\), and \( f'(x) = -\frac{x^{p-1}(1-p\ln x)}{x^{2p}} \) < 0 if \( x > e^{1/p} \), so that \( f \) is eventually decreasing.

\[
\int_{1}^{\infty} \frac{\ln x}{x^p} \, dx = \lim_{t \to \infty} \left[ \frac{x^{1-p}[(1-p)\ln x - 1]}{(1-p)^2} \right]_1^t
\]

(for \( p \neq 1 \)) \( = \frac{1}{(1-p)^2} \lim_{t \to \infty} t^{1-p}[(1-p)\ln t - 1] + 1 \), which exists whenever \( 1 - p < 0 \) \( \iff p > 1 \). So the series is convergent for \( p > 1 \).

EX.33

\( \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x} \), which is \( p \)-series with \( p = x \). In this case, the domain of \( \zeta \) is the set of real numbers \( x \) such that the series is convergent. That is, \( \zeta(x) \) is defined when \( x > 1 \).

EX.41

\[
\sum_{n=1}^{\infty} n^{-1.001} = \sum_{n=1}^{\infty} \frac{1}{n^{1.001}} \text{ is a convergent } p\text{-series with } p = 1.001 > 1. \text{ And }
\]

\[
R_n \leq \int_{n}^{\infty} x^{-1.001} \, dx = \lim_{t \to \infty} \left[ \frac{x^{-0.001}}{-0.001} \right]_{n}^{t} = \frac{1000}{n^{0.001}}
\]

And we want \( R_n < 5 \times 10^{-9} \iff \frac{1000}{n^{0.001}} < 5 \times 10^{-9} \iff n > (2 \times 10^{11})^{1000} \approx 1.07 \times 10^{11.301} \).

EX.44

(a) From the graph, \( \int_{1}^{n+1} \frac{1}{x} \, dx \) is less than the sum of the areas of the \( n \) rectangles.

So, we have

\[
\int_{1}^{n+1} \frac{1}{x} \, dx = \ln(n+1) < 1 + \frac{1}{2} + \ldots + \frac{1}{3} + \frac{1}{n}
\]

\& \( \ln n < \ln(n+1) < 1 + \frac{1}{2} + \ldots + \frac{1}{3} + \frac{1}{n} \), so \( 0 < 1 + \frac{1}{2} + \ldots + \frac{1}{3} + \frac{1}{n} - \ln n = t_n \).

(b) It is clearly that \( \int_{n}^{n+1} \frac{1}{x} \, dx = \ln(n+1) - \ln n > \frac{1}{n+1} \), so \( t_n - t_{n+1} = [\ln(n+1) - \ln n] - \frac{1}{n+1} > 0 \), and this implies that \( \{t_n\} \) is decreasing.
(c) By (b) & (c), we know that \( \{t_n\} \) is decreasing and \( t_n > 0 \) for all \( n \).

So, \( \{t_n\} \) is a bounded monotonic sequence \( \Rightarrow \{t_n\} \) is convergent.

**EX.45**

\[ b^{\ln n} = e^{\ln n \cdot \ln b} = n^{\ln b} = \frac{1}{n^{-\ln b}}. \]

So, \( \sum_{n=1}^{\infty} b^{\ln n} = \sum_{n=1}^{\infty} \frac{1}{n^{-\ln b}} \) is the \( p \)-series, which is convergent for all \( b \) such that \( -\ln b > 1 \Leftrightarrow b < e^{-1} \) (with \( b > 0 \)).

**EX.46**

Let \( s_n = \sum_{i=1}^{n} (\frac{c}{i} - \frac{1}{i+1}) = (\frac{c}{1} - \frac{1}{2}) + (\frac{c}{2} - \frac{1}{3}) + (\frac{c}{3} - \frac{1}{4}) + \cdots + (\frac{c}{n} - \frac{1}{n+1}) \)

\[ = \frac{c}{1} + \frac{c-1}{2} + \frac{c-1}{3} + \cdots + \frac{c-1}{n} - \frac{1}{n+1} = c + (c-1)(\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n}) - \frac{1}{n+1} \]

Thus, \( \sum_{n=1}^{\infty} (\frac{c}{n} - \frac{1}{n+1}) = \lim_{n \to \infty} s_n = \lim_{n \to \infty} [c + (c-1) \sum_{i=2}^{n} \frac{1}{i} - \frac{1}{n+1}], \)

the limit exists only if \( c - 1 = 0 \), so the series is convergent only if \( c = 1 \).