Section 11.2 Series

EX.15
(a) \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} \frac{2n}{3n+1} = \frac{2}{3} \), so \( \{a_n\} \) is convergent.

(b) \( \therefore \lim_{n \to \infty} a_n = \frac{2}{3} \neq 0 \), so \( \sum_{n=1}^{\infty} a_n \) is divergent.

EX.33
\[ \therefore \lim_{n \to \infty} a_n = \lim_{n \to \infty} \sqrt[n]{2} = \lim_{n \to \infty} 2^{\frac{1}{n}} = 2^0 = 1 \neq 0 \], so \( \sum_{n=1}^{\infty} \sqrt[n]{2} \) is divergent.

EX.41
\[ \sum_{n=1}^{\infty} \frac{1}{e^n} \] is a geometric series with ratio \( r = \frac{1}{e} \), and \( |r| = \frac{1}{e} < 1 \).

So, \( \sum_{n=1}^{\infty} \frac{1}{e^n} = \frac{\frac{1}{e}}{1 - \frac{1}{e}} = \frac{\frac{1}{e}}{\frac{e-1}{e}} = \frac{1}{e-1} \). And by example 7, we know that
\[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1 \]. Thus, \( \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \frac{1}{e-1} + 1 = \frac{e}{e-1} \)

EX.48

Let \( s_n = \sum_{i=2}^{n} \left( \frac{1}{(i-1)(i+1)} \right) = \sum_{i=2}^{n} \left( \frac{1}{i-1} - \frac{1}{i+1} \right) \)
\[ = \frac{1}{2} \left( \frac{1}{1} - \frac{2}{3} + \frac{1}{3} \right) + \left( \frac{1}{4} - \frac{2}{7} + \frac{1}{7} \right) + \left( \frac{1}{8} - \frac{2}{11} + \frac{1}{11} \right) + \cdots \]
\[ + \left( \frac{1}{n-3} - \frac{2}{n-2} + \frac{1}{n-2} \right) + \left( \frac{1}{n-2} - \frac{2}{n-1} + \frac{1}{n-1} \right) + \left( \frac{1}{n-1} - \frac{2}{n} + \frac{1}{n} \right) \]
\[ = \frac{1}{2} \left( \frac{1}{1} - \frac{2}{3} + \frac{1}{n-2} + \frac{1}{n-1} \right) = \frac{1}{2} - \frac{1}{2n} + \frac{1}{2n+2} \]
Thus, \( \sum_{n=2}^{\infty} \frac{1}{n^2-n} = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \left( \frac{1}{4} - \frac{1}{2n} + \frac{1}{2n+2} \right) = \frac{1}{4} \)

EX.49
(a) Many people may guess that \( x < 1 \). But in fact, \( x = 1 \).

(b) \( x = 0.99999... = \frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \frac{9}{10000} + \cdots = \sum_{n=1}^{\infty} \frac{9}{10^n} \), which is a geometric series with \( a_1 = 0.9 \) and \( r = \frac{1}{10} \). Its sum is \( \frac{0.9}{1-0.1} = 1 \), so \( x = 1 \).
EX.70

(a) The residual concentration just before the \((n+1)\)st injection is
\[
De^{-aT} + De^{-a2T} + De^{-a3T} + \ldots + De^{-anT} = \frac{De^{-aT(1-e^{-anT})}}{1-e^{-aT}}
\]

(b) The limiting pre-injection concentration is
\[
\lim_{n \to \infty} \frac{De^{-aT(1-e^{-anT})}}{1-e^{-aT}} = \frac{D}{e^{aT-1}}
\]

(c) \(\frac{D}{e^{aT-1}} \geq C \Rightarrow D \geq C(e^{aT} - 1)\), so the minimal dosage is \(D = C(e^{aT} - 1)\).

EX.71

(a) \(S_n = D + Dc + Dc^2 + \ldots + Dc^{n-1} = \frac{D(1-c^n)}{1-c}\)

(b) \(\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{D(1-c^n)}{1-c} = \frac{D}{1-c} \lim_{n \to \infty} (1 - c^n) = \frac{D}{1-c} (\because 0 < c < 1 \Rightarrow \lim_{n \to \infty} c^n = 0)\)
\[
= \frac{D}{s} = kD. \text{ If } c = 0.8, \text{ then } s = 0.2 \Rightarrow k = \frac{1}{s} = 5
\]

EX.76

The area between \(y = x^{n-1}\) and \(y = x^n\) for \(0 \leq x \leq 1\) is
\[
\int_0^1 (x^{n-1} - x^n) \, dx = \left[ \frac{x^n}{n} - \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{n} - \frac{1}{n+1} = \frac{1}{n(n+1)}
\]
we can see as \(n \to \infty\), the sum of the areas between the successive curves approaches the area of the unit square.
So, \(\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1\).

EX.80

Suppose \(\sum_{n=1}^{\infty} a_n\) is convergent, then \(\lim_{n \to \infty} a_n = 0 \Rightarrow \lim_{n \to \infty} \frac{1}{a_n} \neq 0\),
\(\therefore \sum_{n=1}^{\infty} \frac{1}{a_n}\) is divergent.

EX.83

Suppose that \(\sum (a_n + b_n)\) and \(\sum a_n\) are convergent. Then by Theorem 8(iii),
\(\sum [(a_n + b_n) - a_n] = \sum b_n\) is also convergent. But this is a contradiction.
\(\therefore \sum (a_n + b_n)\) is divergent.
EX.84

No. For example, let \( a_n = n \) & \( b_n = -n \). Then both \( \sum a_n \) and \( \sum b_n \) are divergent, but \( \sum (a_n + b_n) = \sum 0 \) is convergent to 0.

EX.88

(a) If \( a_1 = 1, a_2 = 2 \), then the limit seem to be \( \frac{5}{3} \).

If \( a_1 = 2, a_2 = 3 \), then the limit seem to be \( \frac{8}{3} \).

If \( a_1 = 4, a_2 = 1 \), then the limit seem to be 2.

If \( a_1 = 1, a_2 = 4 \), then the limit seem to be 3.

So, we would guess that the limit is \( \frac{a_1 + 2a_2}{3} \).

(b) \( a_n+1 - a_n = \frac{1}{2} (a_n + a_{n-1}) - a_n = -\frac{1}{2} (a_n - a_{n-1}) = \cdots = (-\frac{1}{2})^{n-1} (a_2 - a_1) \).

And \( a_n = a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) = a_1 + \sum_{k=1}^{n-1} (a_{k+1} - a_k) = a_1 + \sum_{k=1}^{n-1} (-\frac{1}{2})^{k-1} (a_2 - a_1) \).

So, \( \lim_{n \to \infty} a_n = a_1 + (a_2 - a_1) \sum_{k=1}^{\infty} (-\frac{1}{2})^{k-1} = a_1 + (a_2 - a_1)[\frac{1}{1 - (-1/2)}] = a_1 + 2a_2 \frac{3}{2} \).

EX.89

(a) \( s_1 = \frac{1}{1/2} = \frac{1}{2}, s_2 = \frac{1}{2} + \frac{1}{1 \cdot 2 \cdot 3} = \frac{5}{6}, s_3 = \frac{5}{6} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} = \frac{23}{24}, s_4 = \frac{23}{24} + \frac{1}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} = \frac{119}{120} \)

The denominator are \( (n + 1)! \), so we guess \( s_n = \frac{(n+1)!-1}{(n+1)!} \).

(b) For \( n = 1 \), \( s_1 = \frac{1}{2} = \frac{2!-1}{2!} \), it holds for \( n = 1 \). Assume \( s_k = \frac{(k+1)!-1}{(k+1)!} \). Then

\[
s_{k+1} = \frac{(k+1)!-1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+2)!-(k+2)+1}{(k+2)!} = \frac{(k+2)!-1}{(k+2)!} \]

Thus, it holds for \( n = k + 1 \). So our guess is correct by induction.

(c) \( \sum_{n=1}^{\infty} \frac{n}{(n+1)!} = \lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{(n+1)!-1}{(n+1)!} = \lim_{n \to \infty} [1 - \frac{1}{(n+1)!}] = 1 \)