1031微乙01-05班期中考解答和評分標準

1. (15%) 在曲線 $y^3+xy^2+x^2y-2x^3=1$ 上，用隱微分求 (a)$\frac{dy}{dx}$ 之公式及求 (b)在點 $x=1$, $y=1$ 處之 $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ 之值。

Solution:
首先對$y^3+xy^2+x^2y-2x^3=1$做隱微分:

$$(y^3)'=y^2y'$$

(2 分)

$$(xy^2)'=y^2+2xyy'$$

(4 分)

$$(x^2y)'=2xy+x^2y'$$

(4 分)

$$(-2x^3)'=-6x^2$$

(1 分)

得到:

$$y^2y'+y^2+2xyy'+2xy+x^2y'-6x^2=0$$

即:

$$y'=\frac{6x^2-y^2-2xy}{3y^2+2xy+x^2}$$

(1 分)

在$(x,y)=(1,1)$得:

$$y'=\frac{1}{2}$$

(1 分)

接著對$y^2y'+y^2+2xyy'+2xy+x^2y'-6x^2=0$再微一次得到:

$$(3y^2+2xy+x^2)y''+(6yy'+2y+2xy'+2x)y'=12x-2yy'-2xy-2xy'$$

在$(x,y)=(1,1)$得:

$$y''=\frac{2}{3}$$

(2 分)

2. (12%) 求落在 $y^2 = 2x$ 抛物線上最靠近 $(1,4)$ 的點。

Solution:

Let the point be $(\frac{y^2}{2}, y)$ which is closest to (1,4). (2%)

It must be a critical point of $H(y) = (\frac{y^2}{2} - 1)^2 + (y-4)^2$ (2%)

$H'(y) = 2(\frac{y^2}{2} - 1)(y) + 2(y-4) = y^3 - 8$ (4%)

Let $H'(y) = y^3 - 8 = 0$,we get $y = 2$ and thus $x = 2$. (2%)

Since $H'(y) > 0(H(y)$ increasing) as $y > 2$ and $H'(y) < 0(H(y)$ decreasing) as $y < 2$,
$y = 2$ must be the absolute minimal point. (2%)

$\therefore$ The closest point is $(x, y) = (2, 2)$

*If you didn't explain why the minimum occurs, you won't get the last 2 points.

3. (21%) 設 $y=f(x)=\frac{x^3}{1+x^2}$。回答下列問題，填入空格並給出**理由**(含計算)。若所求該項**不存在**，請填"**無**"。

(a) 函數遞升之區間為____________________。(5%)

極大點為 $(x,y)=$ ____________________，極小點為$(x,y)=$ ____________________(共 2%)。
理由:

(b) 函數凹向上之區間為____________________，函數凹向下之區間為____________________(共 5%)。

反曲點為 $(x,y)=$ ____________________(2%)。**理由:**

(c) 漸近線為____________________(3%)。**理由:**

(d) 畫函數圖，標明升降處、上凹下凹處、極大極小點、反曲點及漸近線。(若有的話)(4%)

Solution:

(a) $f'(x)=\frac{3x^2(1+x^2)-2x\cdot x^3}{(1+x^2)^2}=\frac{x^2(x^2+3)}{(1+x^2)^2}\geq 0$,$\forall x\in\mathbf{R}$. (4 pts)
so, $f(x)$ is increasing on $(-\infty,\infty)$ (1 pt),
and $f(x)$ has no local max or min. (2 pts)

(b) $f''(x)=\frac{(4x^3+6x)(1+x^2)^2-2(1+x^2)\cdot 2x\cdot(x^4+3x^2)}{(1+x^2)^4}=\frac{-2x(x^2-3)}{(1+x^2)^3}=0\Leftrightarrow x=0,\sqrt{3},-\sqrt{3}$. (1 pt)
And since

$$f''(x)\geq 0\Leftrightarrow x\in(0,\sqrt{3})\cup(-\infty,-\sqrt{3})$$

So, $f(x)$ is concave upward on $(0,\sqrt{3})$ and $(-\infty,-\sqrt{3})$. (2 pts)

$$f''(x)\leq 0\Leftrightarrow x\in(\sqrt{3},\infty)\cup(-\sqrt{3},0)$$

So, $f(x)$ is concave downward on $(\sqrt{3},\infty)$ and $(-\sqrt{3},0)$. (2 pts)
Inflection points: $(0,0)$, $(\sqrt{3},\frac{3\sqrt{3}}{4})$, $(-\sqrt{3},\frac{-3\sqrt{3}}{4})$ (2 pts)

(c) 方法一:

(1)垂直漸近線：

因為 $f(x):R\to R$ 所以沒有垂直漸近線

(2)水平漸近線：

因為 $\lim_{x\to\pm\infty}f(x)=\pm\infty$

所以沒有水平漸近線

(3)斜漸近線：

$f(x)=\frac{x^3}{1+x^2}=\frac{x(1+x^2)-x}{1+x^2}=x+\frac{-x}{1+x^2}$

$\Rightarrow\lim_{x\to\pm\infty}(f(x)-x)=\lim_{x\to\pm\infty}(\frac{-x}{1+x^2})=0$ (3%)

所以斜漸近線為 $y=x$ □

方法二:

(1)垂直漸近線:

同方法一

(2)水平漸近線

同方法一

(3)斜漸近線

令漸近線為 $y = mx + k$

$\Rightarrow \lim_{x\to\pm\infty}(f(x) - y) = \lim_{x\to\pm\infty}(f(x) - (mx + k)) = \lim_{x\to\pm\infty}(f(x) - mx - k) = 0$

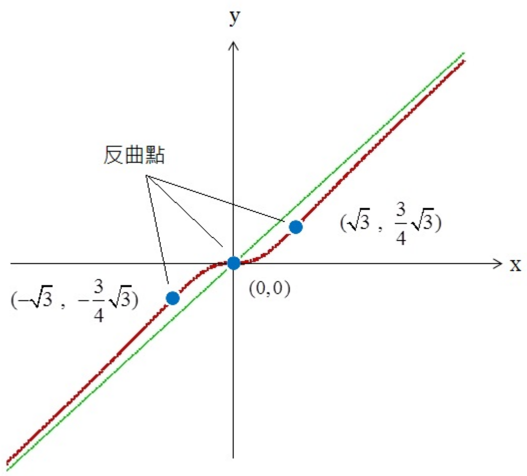
$m = \lim_{x\to\pm\infty}(\frac{f(x)}{x} - \frac{k}{x}) = \lim_{x\to\pm\infty} \frac{f(x)}{x} - \lim_{x\to\pm\infty} \frac{k}{x} = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{x^2}{1+x^2} = 1$ (2%)

$k = \lim_{x\to\pm\infty}(f(x) - mx) = \lim_{x\to\pm\infty}(f(x) - x) = \lim_{x\to\pm\infty}(\frac{x^3}{1+x^2} - \frac{x(1+x^2)}{1+x^2})$

$= \lim_{x\to\pm\infty}(\frac{x^3}{1+x^2} - \frac{x+x^3)}{1+x^2}) = \lim_{x\to\pm\infty} \frac{-x}{1+x^2} = 0$ (1%)

所以漸近線為 $y = x$

(d) 畫函數圖，標明升降處、上凹下凹處、極大極小點、反曲點及漸近線。(若有的話)(4%)



畫出圖形 (1%)

三個反曲點各 (1%)，共 (3%)

其他錯誤，如坐標軸沒標示 x, y、凹向錯誤、無漸近線、點的值錯誤、點的位置錯誤、圖形無趨近漸近線...等，會斟酌扣分。

4. (10%) (a) 求 $(1+x)^{1/3}$ 在 $x=0$ 處的線性逼近公式。

(b) 用(a)估計 $\sqrt[3]{8.03}$ 之值。

Solution:

(a) (5 points)

Let $f(x)=(1+x)^{\frac{1}{3}}, \quad f'(x)=\frac{1}{3}(1+x)^{-\frac{2}{3}}$

$f(0)=1, \quad f'(0)=\frac{1}{3}$

$f(x)\approx f(0)+f'(0)x=1+\frac{1}{3}x$

0r

Let $f(x)=x^{\frac{1}{3}}, \quad f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$

$f(1)=1, \quad f'(1)=\frac{1}{3}$

$f(x)\approx f(1)+f'(1)x=1+\frac{1}{3}x$

(b) (5 points)

$$\begin{aligned}(8.03)^{\frac{1}{3}} &= 2(1+\frac{3}{800})^{\frac{1}{3}}\\ &\approx 2(1+\frac{1}{3}\times\frac{3}{800}), \quad (by(a))\\ &= 2(1+\frac{1}{800})\\ &= \frac{801}{400}\end{aligned}$$

If you solve this problem without (a) , you will get 3 points.

5. (12%) 設 $f(x) = 3x - \sin x$

(a) 說明 $f(x)$ 是遞增函數。

(b) 令 $g(x)$ 是 $f(x)$ 之反函數，求 $g'(0) =?$

Solution:

(a) (6%)
since -1 ≤ cos x ≤ 1 (1%)

$$f'(x) = 3 - \cos x > 0 \ (5\%)$$

hence f(x) is increasing
(b) (6%)
By derivative $f(g(x)) = x$

$$f'(g(x)g'(x) = 1$$

$$g'(0) = \frac{1}{f'(g(0))}$$

$f(0) = 0$ gives

$$0 = g(f(0)) = g(0)$$

Hence

$$g'(0) = \frac{1}{f'(0)} = \frac{1}{2}$$

沒寫到$f(0) = 0$或$g(0) = 0$ 扣一分
答案對但過程有小錯或不完整會扣1到3分不等
寫到一半沒寫完會拿1到2分
直接寫$g'(0) = \frac{1}{f'(0)}$拿1分

6. (20%) 求 y'.

(a) $y = 3^{\cos x}$

(b) $y = \tan^{-1}(1 + x^2)$

(c) $y = \ln(x + \sqrt{1 + x^2})$

(d) $y = \frac{\cos x}{1+\sin x}$

Solution:

(a) $y = 3^{\cos x}$ $[5pts]$

令$f(x) = 3^x$, $g(x) = \cos x$, 因此 $y = f(g(x))$

$$\text{其中} f'(x) = \ln 3 \cdot 3^x \qquad [2pts]$$

$$y' = f'(g(x)) \cdot g'(x) \qquad [1pts]$$

$$= \ln 3 \cdot 3^{\cos x} \cdot (-\sin x) \qquad [2pts]$$

(b) $y = \tan^{-1}(1 + x^2)$ $[5pts]$

令$f(x) = \tan^{-1} x$, $g(x) = 1 + x^2$, 因此 $y = f(g(x))$

$$\text{其中} f'(x) = \frac{1}{1 + x^2} \qquad [2pts]$$

$$y' = f'(g(x)) \cdot g'(x) \qquad [1pts]$$

$$= \frac{2x}{1 + (1 + x^2)^2} \qquad [2pts]$$

(c) $y = \ln(x + \sqrt{1 + x^2})$ $[5pts]$

令$f(x) = \ln x$, $g(x) = x + \sqrt{1 + x^2}$, 因此 $y = f(g(x))$

$$\text{其中} f'(x) = \frac{1}{x} \qquad [2pts]$$

$$y' = f'(g(x)) \cdot g'(x) \qquad [1pts]$$

$$= \frac{1}{\sqrt{1 + x^2}} \qquad [2pts]$$

(d) $y = \frac{\cos x}{1+\sin x}$ $[5pts]$

令$f(x) = \cos x$, $g(x) = 1 + \sin x$, 因此 $y = \frac{f(x)}{g(x)}$

經由leibniz法則:

$$y' = \frac{f'(x)\, g(x) - g'(x)\, f(x)}{g^2(x)} \qquad [2pts]$$

$$= \frac{-1}{1 + \sin x} \qquad [3pts]$$

7. (10%) 求極限

(a) $\lim_{x\to 1^+} \frac{\ln x}{x-1}$

(b) $\lim_{x\to 2} \frac{\sqrt{x^3+x^2-8}-2}{x-2}$

Solution:

(a)(5 pts in total)

Let $h = x-1$, $\lim_{x\to 1^+} \frac{\ln x}{x-1} = \lim_{h\to 0^+} \frac{\ln(h+1)}{h} = \lim_{h\to 0^+} \ln(1+h)^{\frac{1}{h}} = \ln e = 1$.

*With some but insufficient reasons, get 1 pt.
*Right in general but with some minor mistakes, get 4 pts.
*Use L'Höpital's rule directly without any attempt to proof, get 0 pt.

(b)(5 pts in total)

let $f(x) = \sqrt{x^3+x^2-8}$, $\lim_{x\to 2} \frac{\sqrt{x^3+x^2-8}-2}{x-1} = \lim_{x\to 2} \frac{f(x)-f(2)}{x-2} = f'(2) = \left(\frac{1}{2}\frac{3x^2+2x}{\sqrt{x^3+x^2-8}}\right)\Big|_{x=2} = \frac{1}{2} \times \frac{16}{2} = 4$.

*Get 4 pts if it is right in general but with some computational mistakes.
*Any other sufficient simplifications but didn't progress to the last would get 3 pts.
*Get some progress but still have a huge gap between the answer or with some huge mistakes, get 1 pt.
*Use L'Höpital's rule directly without any attempt to proof, get 0 pt.