1031微乙01-05班期末考解答和評分標準

1. (15%) (a) (12%) 求 $\int \frac{dx}{x-1}$, $\int \frac{dx}{(x-1)^2}$, $\int \frac{xdx}{x^2+1}$ 及 $\int \frac{dx}{x^2+1}$。(各3分)

   (b) (3%) 求實數 A, B, C, D 使得下列成立:

$$\frac{4x-2x^2}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

   以此求出 $\int \frac{4x-2x^2}{(x-1)^2(x^2+1)}dx$.

Solution:

(a).

$\int \frac{1}{x-1}dx = ln|x-1| + C$,(let $x-1=u$)

$\int \frac{1}{(x-1)^2}dx = -(x-1)^(-1) + C$,(let $x-1=u$)

$\int \frac{x}{x^2+1}dx = \frac{1}{2}ln(x^2+1) + C$,(let $x^2+1=u$)

$\int \frac{1}{x^2+1}dx = arctan(x) + C$,(let $x = tan\theta$)

每小題3分，如有錯誤會酌量扣分，e.g.:忘記 +C 扣一分，同類型錯誤不反覆扣。

(b).

$$\begin{aligned} 4x-2x^2 &= A(x-1)(x^2+1) + B(x^2+1) + Cx(x-1)^2 + D(x-1)^2 \\ &= (A+C)x^3 + (-A+B-2C+D)x^2 + (A+C-2D)x + (-A+B+D) \end{aligned}$$

$\Rightarrow$

$$\begin{cases} A+C=0 \\ -A+B-2C+D=-2 \\ A+C-2D=4 \\ -A+B+D=0 \end{cases}$$

$\Rightarrow A=-1, B=1, C=1, D=-2$

$$\begin{aligned} \int \frac{4x-2x^2}{(x-1)^2(x^2+1)}dx &= \int \frac{A}{x-1}dx + \int \frac{B}{(x-1)^2}dx + \int \frac{Cx+D}{x^2+1}dx \\ &= \int \frac{-1}{x-1}dx + \int \frac{1}{(x-1)^2}dx + \int \frac{x-2}{x^2+1}dx \\ &= -ln|x-1| - (x-1)^(-1) + \frac{1}{2}ln(x^2+1) - 2arctan(x) + C \end{aligned}$$

求出 A,B,C,D 得兩分，將式子積出得全部分數(共3分)，如有錯誤酌量扣分，如果式子的錯誤根源出在(a)，則不反覆扣。

2. (10%) 導出 $f(x) = \cos^{-1}x$ 在 $x=0$ 處之泰勒展式。寫出一般項，並須計算。(可用 $\mathbf{C}_n^m$ 表示)

Solution:

$$\frac{\mathrm{d}}{\mathrm{d}x}cos^{-1}x = \frac{-1}{\sqrt{1-x^2}} = -(1+(-x^2))^{\frac{-1}{2}} \quad (3\%)$$

By 二項式定理

$$\begin{aligned} -(1+(-x^2))^{\frac{-1}{2}} &= -(C_0^{-\frac{1}{2}} + C_1^{-\frac{1}{2}}(-x^2) + \cdots + C_n^{-\frac{1}{2}}(-x^2)^n + \cdots) \\ &= -\sum_{n=0}^{\infty} C_n^{-\frac{1}{2}}(-x^2)^n \\ &= -\sum_{n=0}^{\infty} (-1)^n C_n^{-\frac{1}{2}} x^{2n} \\ &= \sum_{n=0}^{\infty} (-1)^n C_n^{-\frac{1}{2}} x^{2n} \quad (3\%) \end{aligned}$$

積分回去

$$
\begin{aligned}
cos^{-1}x &= \int \sum_{n=0}^{\infty}(-1)^{n+1}C_n^{-\frac{1}{2}}x^{2n}\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}C_n^{-\frac{1}{2}}\int x^{2n}\,\mathrm{d}x \\
&= \sum_{n=0}^{\infty}(-1)^{n+1}C_n^{-\frac{1}{2}}\frac{1}{2n+1}x^{2n+1}+c \quad (c = cos^{-1}0 = \frac{\pi}{2}) \quad (3\%)
\end{aligned}
$$

一般項 $= \frac{\pi}{2} + \sum_{n=0}^{\infty}(-1)^{n+1}C_n^{-\frac{1}{2}}\frac{1}{2n+1}x^{2n+1} \quad (1\%)$

註1：把$C_n^{-\frac{1}{2}}$展開後答案也可以寫成$\frac{\pi}{2} - \sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{(2n)!}{2^{2n}(n!)^2}x^{2n+1}$

註2：如果寫成定積分的形式，因為$cos^{-1}1 = 0$，所以必須要寫成從 1 積到 x

3. (10%) 求 $\int x^n \ln x dx,\ n \geq 0.$

Solution:

$$
\begin{aligned}
\int x^n \ln x dx &= \int \frac{\ln x}{n+1}dx^{n+1} \\
&= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^{n+1}d\ln x \\
&= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^{n}dx \\
&= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{(n+1)^2}x^{n+1} + c
\end{aligned}
$$

where c is a constant and $n \geq 0$.

評分標準:

(1) 未加常數扣一分;

(2) 沒有計算過程，直接畫表格然後寫答案得0分;

(3) 若算出 $n = 1, 2$ 之後猜出通式得兩分。通式必須用數學歸納法證一次。

4. (8%) 設 R 為 $y = \frac{\sin x}{x}$, x 軸, $x = \frac{\pi}{6}$ 及 $x = \frac{\pi}{2}$ 所圍成之區域。求 R 繞 y 軸旋轉所產生之體積。

Solution:

Use Shell Method:

$$
\begin{aligned}
V &= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\pi x f(x)dx \\
&= \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} 2\pi \sin x dx \\
&= -2\pi \cos x\Big|_{\frac{\pi}{6}}^{\frac{\pi}{2}} \\
&= -2\pi(0 - \frac{\sqrt{3}}{2}) = \sqrt{3}\pi
\end{aligned}
$$

評分標準:

(1) 使用殼形法(Shell method)但積分式 $2\pi x f(x)$ 寫成 $2\pi f(x)$ 得0分;

(2) $\cos\frac{\pi}{2} = 0$，$\cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$，錯一個扣一分;

（3）本題為定積分，對答案增加常數者扣一分。

（4）積分範圍寫反，得到答案 $-\sqrt{3}\pi$ 者，最高得五分。

5. (10%) 將 $y = x - x^2$ 與 $y = 0$ 所圍之區域繞 x 軸旋轉，求所得之旋轉體之體積。

Solution:

$y = x - x^2 = 0 \Leftrightarrow x = 0$ or 1 (3 pts)

$V = \int_0^1 \pi(x-x^2)^2 dx$ (6 pts)

$= \pi \int_0^1 (x^4 - 2x^3 + x^2) dx = \pi(\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3})|_0^1 = \frac{\pi}{30}$(1 pt)

6. (10%) 求曲線 $y = \frac{x^5}{6} + \frac{1}{10x^3}$, $1 \le x \le 2$ 之長度。

Solution:

$$\begin{aligned}
\text{所求} &= \int_1^2 \sqrt{1 + (\frac{dy}{dx})^2}\, dx && (2\%) \\
&= \int_1^2 \sqrt{1 + \left(\frac{5}{6}x^4 - \frac{3}{10x^4}\right)^2}\, dx && (2\%) \\
&= \int_1^2 \sqrt{1 + \frac{25}{36}x^8 - \frac{1}{2} + \frac{9}{100x^8}}\, dx \\
&= \int_1^2 \sqrt{\frac{25}{36}x^8 + \frac{1}{2} + \frac{9}{100x^8}}\, dx \\
&= \int_1^2 \sqrt{\left(\frac{5}{6}x^4 + \frac{3}{10x^4}\right)^2}\, dx \\
&= \int_1^2 \left(\frac{5}{6}x^4 + \frac{3}{10x^4}\right) dx && (2\%) \\
&= \left(\frac{x^5}{6} + \frac{1}{10x^3}\right)\Bigg|_1^2 && (2\%) \\
&= 5\frac{61}{240} \left(\text{ or } \frac{1261}{240}\right) && (2\%)
\end{aligned}$$

7. (9%) 求由 $x^2 = 2y$ 及 $y - x = 4$ 所圍成之有限區域的面積。

Solution:

首先解兩條曲線的交點:

$$y = \frac{x^2}{2}$$
$$y = x + 4$$

得到

$$x^2 - 2x - 8 = 0$$
$$x = -2, 4$$

(各1 分)

因在[-2,4]之間有

$$x+4 \geq \frac{x^2}{2}$$

面積為:

$$A = \int_{-2}^{4} x+4-\frac{x^2}{2} dx$$

(5分)

得到:

$$A = \frac{x^2}{2}+4x-\frac{x^3}{6}\Big|_{-2}^{4} = 18$$

(2分)

8. (8%) 計算極限: $\lim_{x\to 0}(\cos x)^{\frac{1}{x^2}}$.

Solution:

利用:

$$(cosx)^{\frac{1}{x^2}} = e^{ln((cosx)^{\frac{1}{x^2}})} = e^{\frac{1}{x^2}ln(cosx)}$$

有:

$$\lim_{x\to 0}(cosx)^{\frac{1}{x^2}} = \lim_{x\to 0} e^{\frac{1}{x^2}ln(cosx)} = e^{\lim_{x\to 0}\frac{1}{x^2}ln(cosx)}$$

(1分)

故計算:

$$\lim_{x\to 0}\frac{1}{x^2}ln(cosx)$$

則由羅畢達法則得:

$$\lim_{x\to 0}\frac{1}{x^2}ln(cosx) = \lim_{x\to 0}\frac{\frac{-sinx}{cosx}}{2x}$$

(分子分母微分正確各2分)

$$= \lim_{x\to 0} -\frac{1}{2}\frac{sinx}{x}\frac{1}{cosx} = -\frac{1}{2}(\lim_{x\to 0}\frac{sinx}{x})(\lim_{x\to 0}\frac{1}{cosx}) = -\frac{1}{2}$$

(2分)

得到答案:

$$\lim_{x\to 0}(cosx)^{\frac{1}{x^2}} = e^{-\frac{1}{2}}$$

(1分)

9. (12%) (a) (8%) 寫出 $\ln(1-x)$ 及 $\tan^{-1}x$ 在 $x=0$ 之泰勒展式。寫出一般項，不須計算。(各4分)

(b) (4%) 計算極限: $\lim_{x\to 0}\dfrac{\Big(\ln(1-x)\Big)\cdot\Big(\tan^{-1}x\Big)+x^2+\frac{x^3}{2}}{x^5}$ (提示. 利用 (a))

Solution:

(a)(8 points)

$$\begin{aligned} ln(1-x) &= \int_0^x \frac{-1}{1-x}dx \\ &= \int_0^x \sum_{k=0}^{\infty} -x^k dx \quad (-1<x<1) \quad (1pt) \\ &= \sum_{k=1}^{\infty} -\frac{1}{k}x^k \quad (3pt) \end{aligned}$$

$$\begin{aligned} tan^{-1}x &= \int_0^x \frac{1}{x^2}dx \\ &= \int_0^x \sum_{k=0}^{\infty}(-1)^k 1+x^{2k}dx \quad (-1<x<1) \quad (1pt) \\ &= \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}(3pt) \end{aligned}$$

整體係數差一負號者扣1分，交替錯誤者扣2分。

(b)(4 points)

$$\begin{aligned} \lim_{x\to 0}\frac{(ln(1-x))(tan^{-1}x)+x^2+\frac{x^3}{2}}{x^5} &= \lim_{x\to 0}\frac{x^2+\frac{x^3}{2}-(x+\frac{x^2}{2}+\frac{x^3}{3}+\frac{x^4}{4}+\ldots)(x-\frac{x^3}{3}+\frac{x^5}{5}+\ldots)}{x^5} \\ &= \lim_{x\to 0}\frac{x^2+\frac{x^3}{2}+(-x^2-\frac{x^3}{2}-\frac{x^5}{12}-\frac{x^6}{45}+\ldots)}{x^5} \\ &= \lim_{x\to 0}\frac{-\frac{x^5}{12}-\frac{x^6}{45}+\ldots}{x^5} \\ &= \frac{-1}{12} \end{aligned}$$

若(a)正確，乘法錯致使結果錯只得3分，極限值與計算所得不合者只得1分。
若(a)錯誤，過程都對得兩分，乘法錯誤但取極限過程正確得1分。

10. (8%) 求 $\int\sqrt{1-x^2}dx.$

Solution:

let $x=\sin\theta$

$$\begin{aligned} \int\sqrt{1-x^2}\mathrm{d}x = \int\cos\theta\mathrm{d}\sin\theta = \int\cos^2\theta\mathrm{d}\theta = \int\frac{1+\cos 2\theta}{2}\mathrm{d}\theta \ (2\%) \\ = \frac{\theta}{2}+\frac{\sin 2\theta}{4}+c = \frac{\theta}{2}+\frac{\sin\theta\cos\theta}{2}+c \ (2\%) \\ = \frac{\arcsin x}{2}+\frac{x\sqrt{1-x^2}}{2}+c \ (4\%) \end{aligned}$$

計算錯誤看情況扣 1 到 2 分, 沒加 c 扣 1 分