1031微甲07-11班期中考解答和評分標準

1. (10%) Find the limit $\lim_{x\to 0}\left(\frac{a+x}{a-x}\right)^{1/x}$, $a>0$.

Solution:

(Method 1) First we observe that

$$\lim_{x\to 0}\left(\frac{a+x}{a-x}\right)^{\frac{1}{x}} = \lim_{x\to 0} e^{\ln\left(\frac{a+x}{a-x}\right)^{\frac{1}{x}}} = \lim_{x\to 0} e^{\frac{1}{x}\ln\left(\frac{a+x}{a-x}\right)} = e^{\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{a+x}{a-x}\right)} \text{ (2 points)}$$

Note that $\frac{1}{x}\ln\left(\frac{a+x}{a-x}\right)$ is of the form $\frac{0}{0}$, hence by L'Hopital's rule (1 point), we have

$$\begin{aligned}
(1)\ \lim_{x\to 0}\frac{1}{x}\ln\left(\frac{a+x}{a-x}\right) &= \lim_{x\to 0}\frac{\ln(a+x)-\ln(a-x)}{x}\\
&= \lim_{x\to 0}\frac{\frac{1}{a+x}(a+x)'-\frac{1}{a-x}(a-x)'}{1} \text{ (2 points)}\\
&= \lim_{x\to 0}\frac{\frac{1}{a+x}-\frac{-1}{a-x}}{1}\\
&= \lim_{x\to 0}\frac{1}{a+x}+\frac{1}{a-x} \text{ (1 points)}\\
&= \frac{2}{a} \text{ (1 points)}
\end{aligned}$$

$$\begin{aligned}
(2)\ \lim_{x\to 0}\frac{1}{x}\ln\left(\frac{a+x}{a-x}\right) &= \lim_{x\to 0}\frac{\ln\left(\frac{a+x}{a-x}\right)}{x}\\
&= \lim_{x\to 0}\frac{\left(\frac{a+x}{a-x}\right)^{-1}\cdot\frac{d}{dx}\left(\frac{a+x}{a-x}\right)}{1} \text{ (2 points)}\\
&= \lim_{x\to 0}\frac{a-x}{a+x}\cdot\frac{(a-x)\cdot 1-(-a)(a+x)}{(a-x)^2}\\
&= \lim_{x\to 0}\frac{a-x+a+x}{(a+x)(a-x)}\\
&= \lim_{x\to 0}\frac{2a}{a^2-x^2} \text{ (1 points)}\\
&= \frac{2}{a} \text{ (1 points)}
\end{aligned}$$

$$\begin{aligned}
(3)\ \lim_{x\to 0}\frac{1}{x}\ln\left(\frac{a+x}{a-x}\right) &= \lim_{x\to 0}\frac{\ln(a+x)-\ln(a-x)}{x}\\
&= \lim_{x\to 0}\frac{\ln(a+x)-\ln a+\ln a-\ln(a-x)}{x} \text{ (2 points)}\\
&= \lim_{x\to 0}\frac{\ln(a+x)-\ln a}{x}+\lim_{x\to 0}\frac{\ln(a-x)-\ln a}{-x}\\
&= \lim_{x\to 0}\frac{\ln(a+x)-\ln a}{x}+\lim_{y\to 0}\frac{\ln(a+y)-\ln a}{y} \text{ (1 points)}\\
&= 2\left[\frac{d}{dt}\ln t\right]_{t=a} = \frac{2}{a} \text{ (1 points)}
\end{aligned}$$

Hence

$$\lim_{x\to 0}\left(\frac{a+x}{a-x}\right)^{\frac{1}{x}} = e^{\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{a+x}{a-x}\right)} = e^{\frac{2}{a}}. \text{ (3 points)}$$

If you serve $\frac{2}{a}$ as the answer and have previously indicated that $e^{\lim_{x\to 0}\frac{1}{x}\ln\left(\frac{a+x}{a-x}\right)}$, you still get the last 3 points.

(Method 2)

First we observe that $\dfrac{a+x}{a-x} = 1 + \dfrac{2x}{a-x}$. (2 points)

$$\lim_{x\to 0}\left(\frac{a+x}{a-x}\right)^{\frac{1}{x}} = \lim_{x\to 0}\left(1+\frac{2x}{a-x}\right)^{\frac{1}{x}}$$

By letting $t = \dfrac{2x}{a-x}$, $t \to 0$ as $x \to 0$. (3 point) Hence

$$\begin{aligned}
\lim_{x\to 0}\left(1+\frac{2x}{a-x}\right)^{\frac{1}{x}} &= \lim_{x\to 0}\left(1+\frac{2x}{a-x}\right)^{\frac{a-x}{2x}\frac{2}{a-x}} \quad \text{(2 points)} \\
&= \lim_{t\to 0}(1+t)^{\frac{1}{t}\cdot\frac{2}{a-x}} = \lim_{t\to 0}\left[(1+t)^{\frac{1}{t}}\right]^{\frac{2}{a-x}}
\end{aligned}$$

Note that we have $\lim\limits_{x\to 0} f(x)^{g(x)} = \lim\limits_{x\to 0} f(x)^{\lim_{x\to 0} g(x)}$ provided the limits exists.

$$\lim_{x\to 0}\left[(1+t)^{\frac{1}{t}}\right]^{\frac{2}{a-x}} = \left[\lim_{t\to 0}(1+t)^{\frac{1}{t}}\right]^{\lim_{x\to 0}\frac{2}{a-x}} = e^{\frac{2}{a}} \text{ (3 points)}$$

(Method 3)

First we let $t = \dfrac{1}{x}$. (2 points)

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{a+x}{a-x}\right)^{\frac{1}{x}} &= \lim_{t\to\infty}\left(\frac{a+\frac{1}{t}}{a-\frac{1}{t}}\right)^{t} = \lim_{t\to\infty}\left(\frac{at+1}{at-1}\right)^{t} \\
&= \lim_{t\to\infty}\left(1+\frac{2}{at-1}\right)^{t} \quad \text{(3 points)} \\
&\approx \lim_{t\to\infty}\left(1+\frac{2}{at}\right)^{t} \quad \text{(2 points)} \\
&= \lim_{t\to\infty}\left(1+\frac{\frac{2}{a}}{t}\right)^{t} = e^{\frac{2}{a}} \quad \text{(3 points)}
\end{aligned}$$

(We should prevent using this unrigorous calculation.)

(Method 4)

By letting $t = \dfrac{1}{x}$. (2 points)

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{a+x}{a-x}\right)^{\frac{1}{x}} &= \lim_{x\to 0}\left(\frac{1+\frac{x}{a}}{1-\frac{x}{a}}\right)^{\frac{1}{x}} = \lim_{x\to 0}\frac{\left(1+\frac{x}{a}\right)^{\frac{1}{x}}}{\left(1-\frac{x}{a}\right)^{\frac{1}{x}}} \quad \text{(3 points)} \\
&= \lim_{t\to\infty}\frac{\left(1+\frac{1/a}{t}\right)^{t}}{\left(1+\frac{-1/a}{t}\right)^{t}} \\
&= \frac{\lim_{t\to\infty}\left(1+\frac{1/a}{t}\right)^{t}}{\lim_{t\to\infty}\left(1+\frac{-1/a}{t}\right)^{t}} \quad \text{(2 points)} \\
&= \frac{e^{1/a}}{e^{-1/a}} = e^{\frac{2}{a}} \quad \text{(3 points)}
\end{aligned}$$

2. (10%) Find the limit $\lim_{x\to 0} \frac{\ln(\cos(ax))}{\ln(\cos(bx))}$, where a, b are constant.

Solution:

We found this problem is an indeterminate form.
Consequently, we think about the L'Hospital's Rule first.
In this rule, there are three assumptions should be satisfied:
(1)The denominator and the numerator should be differentiable on an open interval that contains 0.
(2) The differentiation of the denominator can NOT equal to zero on an open interval that contains 0.

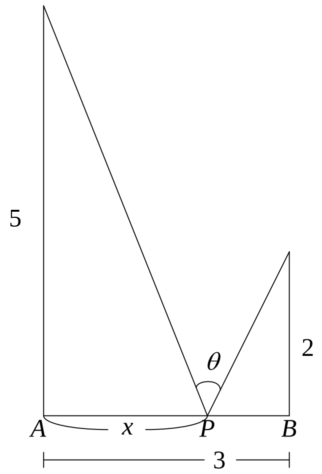
$(\ln\cos(ax))' = \frac{1}{\cos(ax)} \cdot (-\sin(ax)) \cdot a$
$(\ln\cos(bx))' = \frac{1}{\cos(bx)} \cdot (-\sin(bx)) \cdot b$
$(-\sin(ax))' = -\cos(ax) \cdot a$
$(-\sin(bx))' = -\cos(bx) \cdot b$
Therefore, the origimal can be rewritten as follows by using L'Hospital's Rule twice:
$\lim_{x\to 0} \frac{\cos bx}{\cos ax} \cdot \frac{-\cos ax \cdot a}{-\cos bx \cdot b} \cdot \frac{a}{b} = \frac{a^2}{b^2}$.

If students use L'Hospital's Rule first time correctly, and they can get 4 points;
If students use L'Hospital's Rule second time correctly, and they can get another 4 points;
If students calculate the limitation under x approaches to zero correctly, and they can get the other 2 points.

Solution 2
By using L'Hospital's Rule, the original equation can be written as follows:
$\lim_{x\to 0} \frac{\frac{-\sin(ax)}{\cos(ax)} \cdot a}{\frac{-\sin(bx)}{\cos(bx)} \cdot b}$
$= \lim_{x\to 0} \frac{a\tan(ax)}{b\tan(bx)}$
$= \lim_{x\to 0} \frac{a\sec^2(ax) \cdot a}{b\sec^2(bx) \cdot b}$
$= \frac{a^2}{b^2}$.

Other proper methods are permitted to solve this problem.

3. (12%) Choose the point P on the line segment AB so as (a) to maximize the angle θ; (b) to minimize the angle θ.



Solution:

$\theta = \pi - \tan^{-1}(5/x) - \tan^{-1}(\frac{2}{3-x})$ (4pts)

$\theta' = \frac{\frac{5}{x^2}}{1+\frac{25}{x^2}} + \frac{\frac{2}{(3-x)^2}}{1+\frac{4}{(3-x)^2}}$

$= \frac{5}{x^2+25} - \frac{2}{x^2-6x+13}$

$\theta' = 0 \implies x^2 - 10x + 5 = 0$

$x = 5 - 2\sqrt{5}$(since $5 + 2\sqrt{5} > 3$)(4pts)

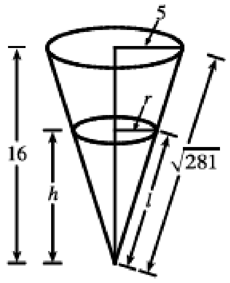
$\theta > 0$ when $x \in (0, 5-2\sqrt{5})$
$\theta < 0$ when $x \in (5-2\sqrt{5}, 3)$

Thus $x = 5 - 2\sqrt{5}$ is a local maximum also a maximum since θ is differentiable on (0,3) (2pts)
Check endpoints for minimum.
$x = 0$ then $\theta = \pi/2 - \tan^{-1}(\frac{2}{3})$
$x = 3$ then $\theta = \pi/2 - \tan^{-1}(\frac{5}{3})$

Since arctangent is increasing ,θ has a minimum when $x = 3$ (2pts)
Answer:(a)$x = 5 - 2\sqrt{5}$(b)$x = 3$

4. (12%) A container in the shape of an inverted cone has height 16 cm and radius 5 cm at the top. It is partially filled with a liquid that oozes(滲出) through the sides at a rate proportional to the area of the container that is in contact with the liquid. If we pour the liquid into the container at a rate of 2 cm^3/min, then the height of the liquid decreases at a rate of 0.3 cm/min when the height is 10 cm. If our goal is to keep the liquid at a constant height of 10 cm, at what rate should we pour the liquid into the container?

- You may need this: The surface area of a cone is $\pi r l$, where r is the radius and l is the slant height(斜高).



Solution:

Let
$V_{pour}(t)$ be the volume of the liquid pours into the container at time t
$V_{ooze}(t)$ be the volume of the liquid oozes through the container at time t
h(t) be the height of the liquid in the container at time t,as h in the figure
r(t) be the radius of the liquid in the container at time t,as r in the figure
V(t) be the volume of liquid in the container at time t

by similarity of triangles : $\dfrac{r(t)}{5} = \dfrac{h(t)}{16} \;\Rightarrow r(t) = \dfrac{5}{16}h(t)$

we also have :
V_0 $(initial\ volume\ of\ the\ liquid)\ +\ V_{pour}(t)\ -\ V_{ooze}(t) = V(t)$
$= \dfrac{1}{3}\pi r^2(t)h(t) = \dfrac{25}{768}\pi h^3(t)...(4\%)$

$\Rightarrow V'_{pour}(t) - V'_{ooze}(t) = V'(t) = \dfrac{25}{256}\pi h^2(t)h'(t)...(4\%)$

notice that : $h'(t_0) = -0.3\ at\ h(t_0) = 10, V'_{pour}(t_0) = 2$

$\Rightarrow 2 - V'_{ooze}(t_0) = \dfrac{25}{256}\pi(10^2)(-0.3) = -\dfrac{375\pi}{128}...(2\%)$

$\Rightarrow V'_{ooze}(t_0) = 2 + \dfrac{375\pi}{128}$

so if $V'_{pour}(t_0) = V'_{ooze}(t_0) = 2 + \dfrac{375\pi}{128}$ (cm^3/min) ...(2%)

then $V'(t_0) = V'_{pour}(t_0) - V'_{ooze}(t_0) = 0$,hence $h'(t_0) = 0$ which is what we want

(If you finish the second part(state the differential equation),then the first 4 points will be included.There is 0 points for the third part if the first two parts are not both correct.If the relation of functions are wrong,there is 0 points for this problem!)

5. (12%) If $xy + e^y = e$,

(a) (3%) find $\frac{dy}{dx}$.

(b) (9%) find the values of y, y' and y'' at the point where $x = 0$.

Solution:

(a) $y + x\frac{dy}{dx} + e^y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx}(x + e^y) = -y \Rightarrow \frac{dy}{dx} = \frac{-y}{x + e^y}$

(b) $x = 0 \Rightarrow e^y = 1 \Rightarrow y = 1$, and $y' + y' + xy'' + e(y')^2 + e^y y'' = 0$, then $y' = \frac{-1}{0+e} = \frac{-1}{e}$ when $x = 0$, then take $y = 0, y' = \frac{-1}{e}$, we get $\frac{-2}{e} + e(\frac{1}{e})^2 + ey'' = 0 \Rightarrow ey'' = \frac{1}{e} \Rightarrow y'' = \frac{1}{e^2}$

6. (10%) Define $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$

(a) (3%) Show that $f(x)$ is continuous at $x = 0$.

(b) (4%) Calculate $f'(x)$ when $x \neq 0$.

(c) (3%) Find $f'(0)$ if it exists.

Solution:

(a) We need to check that $\lim_{x\to 0} f(x) = f(0) = 0$. (2pts)

$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.

(since $\forall x \neq 0$, $-x^2 \leq x^2 \sin(\frac{1}{x}) \leq x^2$ & $\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0$.

Then by squeezing theorem, we have $\lim_{x\to 0} x^2 \sin(\frac{1}{x}) = 0$.) (1pt)

(b) When $x \neq 0$, $f'(x) = 2x \sin \frac{1}{x} + x^2 (\cos \frac{1}{x})(\frac{-1}{x^2}) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$ (4pts)

(c) $f'(0) = \lim_{x\to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \dfrac{x^2 \sin \frac{1}{x} - 0}{x - 0} = \lim_{x\to 0} x \sin \dfrac{1}{x} = 0$. (2pts)

(since $\forall x \neq 0$, $-|x| \leq x \sin(\frac{1}{x}) \leq |x|$ & $\lim_{x\to 0} x = \lim_{x\to 0} -x = 0$.

Then by squeezing theorem, we have $\lim_{x\to 0} x \sin \frac{1}{x} = 0$.) (1pt)

7. (24%) Let $f(x) = \dfrac{x^4}{(1+x)^3}$. Answer the following questions by filling each blank below. Show your work (computations and reasoning) in the space following. Put **None** in the blank if the item asked does **not** exist.

(a) The function is increasing on the interval(s)____________________ and decreasing on the interval(s)____________________(6% total). The local maximal point(s) $(x, y) =$ ________________(2%), The local minimal point(s) $(x, y) =$____________________(2%).

(b) The function is concave upward on the interval(s)____________________ and concave downward on the interval(s)____________________(6% total).

The inflection point(s) $(x, y) =$ ____________________(2%).

(c) The vertical asymptote line(s) of the function is(are) ____________________.

The horizontal asymptote line(s) is(are) ____________________(2%)

(d) Sketch the graph of the function. Indicate, if any, where it is increasing/decreasing, where it concaves upward/downward, all relative maxima/minima, inflection points and asymptotic line(s) (if any).(4%)

Solution:

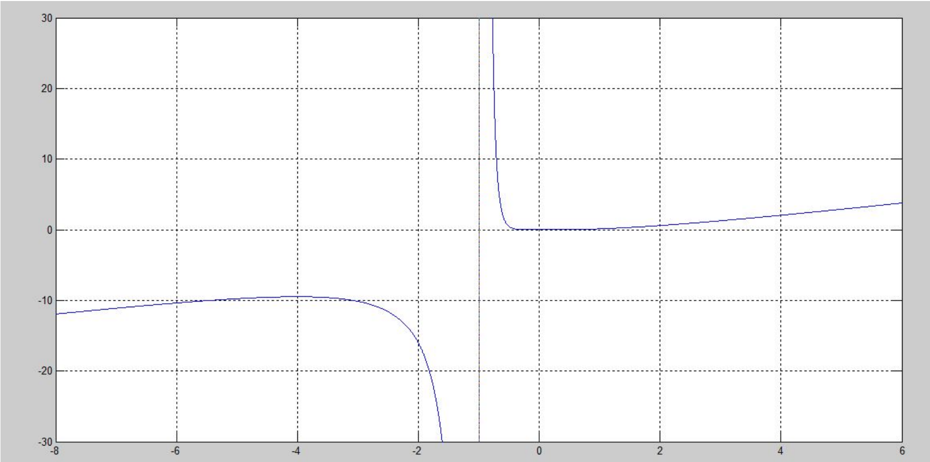
(a) $f'(x) = \dfrac{4x^3(1+x)^3 - 3(1+x)^2x^4}{(1+x)^6} = \dfrac{4x^3(1+x) - 3x^4}{(1+x)^4} = \dfrac{x^3(x+4)}{(1+x)^4}$ (4 points)

$f(x) = 0 \Rightarrow x^3(x+4) = 0 \Rightarrow x = 0$ or 4
$f'(x) > 0$ on $(-\infty, -4) \cup (0, -\infty)$
$\Rightarrow f$ is increasing on $(-\infty, -4) \cup (0, -\infty)$. (1 point)
$f'(x) < 0$ on $(-4, -1) \cup (-1, 0)$
$\Rightarrow f$ is decreasing on $(-4, -1) \cup (-1, 0)$. (1 point)
$\Rightarrow f$ has a local minimum at $(0, 0)$ (2 points)
and a local maximun at $(-4, -\dfrac{256}{27})$. (2 points)

(b) $f''(x) = \dfrac{(3x^2(x+4) + x^3)(1+x)^4 - 4(1+x)^3(x^3(x+4))}{(1+x)^8} = \dfrac{(1+x)(4x^3 + 12x^2) - (4x^4 + 16x^3)}{(1+x)^5}$
$= \dfrac{12x^2}{(1+x)^5}$ (4 points)
$f''(x) > 0 \Rightarrow \dfrac{12x^2}{(1+x)^5} > 0, x \neq 0 \Rightarrow x > -1$, $x \neq 0$.
$f''(x) < 0 \Rightarrow \dfrac{12x^2}{(1+x)^5} < 0 \Rightarrow x < -1$.
f is concave upward on $(-1, \infty)$, (1 point)
and concave downward on $(-\infty, -1)$. (1 point)
$(0, 0)$ is not an inflection point, since $f''(x) > 0$ on $(-1, 0)$ and $(0, \infty)$.
f is not defined at $x = -1$.
f has no inflection point. (2 points)

(c) $\lim_{x \to -1^+} f(x) = \infty$
$\Rightarrow f(x)$ has vertical asymptone $x = -1$. (1 point)
$\lim_{x \to \pm\infty} f(x) = \pm\infty$
$\Rightarrow f(x)$ has no horizontal asymptone. (1 point)

(d) The figure is:


30
20
10
0
-10
-20
-30
-8
-6
-4
-2
0
2
4
6

8. (10%) Consider any point (x_0, y_0), where $x_0 > 0$ and $y_0 > 0$, on the hyperbola $xy = k$, where $k > 0$ is a constant.

(a) (6%) Find the equation of the tangent line at (x_0, y_0).

(b) (3%) Let A and B denote respectively the x-intercept and the y-intercept of the tangent line at (x_0, y_0). Find the area of the triangle enclosed by the origin O and A, B.

(c) (1%) Is the area of $\triangle OAB$ a constant? That is, is the area of $\triangle OAB$ independent of x_0 and y_0?

Solution:

Method 1 (a) To find the tangent line of $xy = k$ at (x_0, y_0) in the first quadrant, we differentiate $xy = k$ implicitly with respect to x to get

$$y + x\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x} \ \textbf{(3\%)} \ \Rightarrow \left.\frac{\mathrm{d}y}{\mathrm{d}x}\right|_{(x_0,y_0)} = -\frac{y_0}{x_0}. \ \textbf{(1\%)}$$

So the tangent line of $xy = k$ at (x_0, y_0) is $y - y_0 = -\frac{y_0}{x_0}(x - x_0)$. **(2%)**

(b) We get x-intercept and y-intercept of the tangent line as follows:

- Let $y = 0$, then $x = 2x_0$, so the coordinates of A is $(2x_0, 0)$. **(1%)**
- Let $x = 0$, then $y = 2y_0$, so the coordinates of B is $(0, 2y_0)$. **(1%)**

Hence the area of $\triangle OAB$ is $\frac{1}{2} \cdot 2x_0 \cdot 2y_0 = 2x_0y_0$. **(1%)**

(c) Yes, the area of $\triangle OAB$ is $2x_0y_0 = 2k$, which is a constant. **(1%)**

The hyperbola $xy = k$ can be written as $y = \frac{k}{x}$.

Method 2 (a) Since $y = \frac{k}{x}$, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{k}{x^2} \ \textbf{(3\%)} \ \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = -\left.\frac{k}{x^2}\right|_{x=x_0} = -\frac{k}{x_0^2}. \ \textbf{(1\%)}$$

So the tangent line of $xy = k$ at (x_0, y_0) is $y - y_0 = -\frac{k}{x_0^2}(x - x_0)$. **(2%)**

(b) We get x-intercept and y-intercept of the tangent line as follows:

- Let $y = 0$, then $x = x_0 + \frac{x_0^2 y_0}{k}$, so the coordinates of A is $(x_0 + \frac{x_0^2 y_0}{k}, 0)$. **(1%)**
- Let $x = 0$, then $y = y_0 + \frac{k}{x_0}$, so the coordinates of B is $(0, y_0 + \frac{k}{x_0})$. **(1%)**

Hence the area of $\triangle OAB$ is $\frac{1}{2}\left(x_0 + \frac{x_0^2 y_0}{k}\right)\left(y_0 + \frac{k}{x_0}\right)$. **(1%)**

(c) Yes, the area of $\triangle OAB$ is $\frac{1}{2}\left(x_0y_0 + k + \frac{x_0^2 y_0^2}{k} + x_0y_0\right) = 2k$, which is a constant. **(1%)**

微分計算錯誤，代點得到錯誤的值，但是有寫出直線方程式(斜率當然有誤)，得 2 分。
因為 (a) 計算錯誤，但有確實找直線和兩軸的交點以及面積，得 2 分。
(c) 寫了 Yes，但是在 (b) 或 (c) 的過程中並未寫出算得之面積為 $2k$(常數)，不給分。