1022微乙01-05班期中考解答和評分標準

1. (12%) 求過曲面 $y - x = 4\arctan(xz)$ 上點 $(1,1,0)$ 的切平面方程式。

Solution:
令 $f(x,y,z) = y - x - 4\tan^{-1}(xz)$

$$\nabla f(x,y,z) = (-1 - 4\frac{z}{1+(xz)^2},\ 1,\ \ -4\frac{x}{1+(xz)^2}) \quad (6\%)$$

$$\nabla f(1,1,0) = (-1,1,-4) \quad (2\%)$$

$$\nabla f(1,1,0)\cdot(x-1, y-1, z-0) = 0 \quad (4\%)$$

2. (12%) 設在點 (x,y) 之溫度為 $T(x,y) = 100e^{-x^2-3y^2}$, 此處 T 以攝氏 °C 計，x, y 以 m(公尺) 計，
   (a) 在點 $(1,-1)$ 處，往那個方向增加最快？
   (b) 此最快增加率（以 °C/m 計）為何？
   (c) 求 方向導數 $\frac{\partial T}{\partial \vec{u}}(1,-1)\Big|_{\vec{u}=(\frac{3}{5},\frac{4}{5})}$.

Solution:
Since $\nabla T = T(-2x,-6y)$ (3pt) $\Rightarrow \nabla T(1,-1) = 200e^{-4}(-1,3)$ (3pt)
(a) in the direction: $\nabla T(1,-1) = 200e^{-4}(-1,3)$ (2pt)
(b) fastest increasing rate: $|\nabla T(1,-1)| = 200e^{-4}\sqrt{10}$ (2pt)
(c) directional derivative: $\nabla T(1,-1)\cdot(\frac{3}{5},\frac{4}{5}) = 360e^{-4}$ (2pt)

3. (12%) 求 $\int_0^1\left(\int_{x^{\frac{1}{6}}}^1 \frac{1}{1+y^7}dy\right)dx$.

Solution:

$$\begin{aligned}
\int_0^1\left(\int_{x^{\frac{1}{6}}}^1 \frac{1}{1+y^7}dy\right)dx &= \int_0^1\left(\int_0^{y^6} \frac{1}{1+y^7}dx\right)dy && (4\,points)\\
&= \int_0^1 \frac{y^6}{1+y^7}dy && (2\,points)\\
&= \frac{1}{7}\int_0^1 \frac{1}{1+y^7}dy^7 \\
&= \frac{1}{7}\ln(1+y^7)\ |_0^1 && (4\,points)\\
&= \frac{1}{7}\ln 2 && (2\,points)
\end{aligned}$$

4. (12%) Ω 由 $2x+3y=0, 3x+y=0$ 與 $x-2y=1$ 所圍成的區域. 計算

$$I = \int\int_\Omega (x-2y)^{3/2}(3x+y)^{1/2}dA.$$

Solution:

Use changing variable, let

$$\begin{cases} u &= x-2y \\ v &= 3x+y \end{cases} \quad (2\%)$$

$$\Rightarrow \begin{cases} x &= \dfrac{-3u+v}{7} \\ y &= \dfrac{u+2v}{7} \end{cases} \quad (1\%)$$

then Jacobian is

$$J = |\begin{vmatrix} -\frac{3}{7} & \frac{1}{7} \\ \frac{1}{7} & \frac{2}{7} \end{vmatrix}| = \frac{1}{7} \quad (4\%)$$

Therefore

$$\begin{aligned} I &= \iint_\Omega (x-2y)^{3/2}(3x+y)^{1/2}dA \quad (4\%) \\ &= \int_0^1 \int_0^u u^{3/2}v^{1/2} \cdot \frac{1}{7}\, dvdu \\ &= \frac{1}{7}\int_0^1 \frac{2}{3}u^{\frac{3}{2}}v^{\frac{3}{2}}\Big|_0^u\, du \\ &= \frac{2}{21}\int_0^1 u^3\, du \\ &= \frac{1}{42} \quad (1\%) \end{aligned}$$

By the way, you can also write

$$I = \int_0^1 \int_v^1 u^{3/2}v^{1/2} \cdot \frac{1}{7}\, dudv = \frac{1}{42}$$

5. (15%) 決定 $z = ye^{-\frac{1}{2}(x^2+y^2)}$ 的極值候選點，並決定其極值性質。

Solution:

Let $f(x,y) = ye^{-\frac{1}{2}(x^2+y^2)}$

$$\begin{aligned} f_x &= -xye^{-\frac{1}{2}(x^2+y^2)} && (1\%) \\ f_y &= (1-y^2)e^{-\frac{1}{2}(x^2+y^2)} && (1\%) \\ f_{xx} &= (x^2y-y)e^{-\frac{1}{2}(x^2+y^2)} && (1\%) \\ f_{yy} &= (y^3-3y)e^{-\frac{1}{2}(x^2+y^2)} && (1\%) \\ f_{xy} &= (xy^2-x)e^{-\frac{1}{2}(x^2+y^2)} = f_{yx} && (1\%) \end{aligned}$$

If we solve $f_x = f_y = 0$, which equal to solve:

$$\begin{aligned} xy &= 0 \\ 1-y^2 &= 0 \end{aligned}$$

and thus the critical points are $(0,1)$ and $(0,-1)$. (4%)

Now let

$$D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$$

$$D(0,1) = \frac{2}{e} > 0,\ f_{xx}(0,1) = -e^{\frac{1}{2}} < 0 \quad \Rightarrow \quad (0,1) \text{ is a local maximum}$$

$$D(0,-1) = \frac{2}{e} > 0,\ f_{xx}(0,1) = e^{\frac{1}{2}} > 0 \quad \Rightarrow \quad (0,-1) \text{ is a local minimum} \quad (6\%)$$

6. (10%) 設 $f(x,y) = x^3y^5 - x^2y - y^3$, $x = x(u,v)$, $y = y(u,v)$.

已知 $x(1,3)=2$, $y(1,3)=1$

且 $\dfrac{\partial x}{\partial u}(1,3)=\dfrac{1}{5}$, $\quad \dfrac{\partial x}{\partial v}(1,3)=\dfrac{1}{2}$

$\dfrac{\partial y}{\partial u}(1,3)=\dfrac{1}{11}$, $\quad \dfrac{\partial y}{\partial v}(1,3)=\dfrac{1}{4}$

求 $\dfrac{\partial f}{\partial u}$ 及 $\dfrac{\partial f}{\partial v}$ 在 $(u,v)=(1,3)$ 的值.

Solution:
We know that

$$\begin{aligned}\frac{\partial f}{\partial u} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial u}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial u} \quad (3\text{pts}),\\ \frac{\partial f}{\partial v} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial v}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial v} \quad (3\text{pts}).\end{aligned}$$

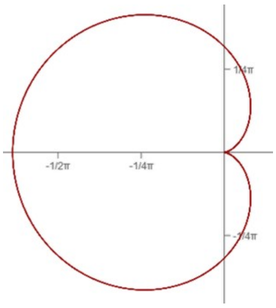
Moreover,

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3x^2y^5-2xy \qquad (1\text{pt}),\\ \frac{\partial f}{\partial y} &= 5x^3y^4-x^2-3y^2 \quad (1\text{pt}).\end{aligned}$$

Thus, when $(u,v)=(1,3)$,

$$\begin{aligned}\frac{\partial f}{\partial u}\Big|_{(u,v)=(1,3)} &= 8\times\frac{1}{5}+33\times\frac{1}{11}=\frac{23}{5} \quad (1\text{pt}),\\ \frac{\partial f}{\partial v}\Big|_{(u,v)=(1,3)} &= 8\times\frac{1}{2}+33\times\frac{1}{4}=\frac{49}{4} \quad (1\text{pt}).\end{aligned}$$

7. (12%) 求 $\iint_\Omega \sqrt{x^2+y^2}dA$, Ω 為心形線 $r=1-\cos\theta$ 的內部。



Solution:
Use polar coordinate, let $x=r\cos\theta$, $y=r\sin\theta$, then we obtain

$$\begin{aligned}\iint_\Omega \sqrt{x^2+y^2}\,dA &= \int_0^{2\pi}\int_0^{1-\cos\theta} r^2\,dr d\theta \quad (4\text{pt})\\ &= \int_0^{2\pi}\frac{(1-\cos\theta)^3}{3}\,d\theta \quad (1\text{pt})\\ &= \int_0^{2\pi}\frac{1}{3}-\cos\theta+\cos^2\theta-\frac{\cos^3\theta}{3}\,d\theta\\ &= \frac{\theta}{3}-\sin\theta-\frac{2+\sin^2\theta}{4}\Big|_0^{2\pi}-\frac{1}{3}\int_0^{2\pi}(1-\sin^2\theta)\,d\sin\theta \quad (1\text{pt},1\text{pt},2\text{pt},1\text{pt})\\ &= \frac{5\pi}{3}-\frac{1}{3}\left[\sin\theta-\frac{\sin^3\theta}{3}\right]_0^{2\pi} \quad (1\text{pt})\\ &= \frac{5\pi}{3} \quad (1\text{pt})\end{aligned}$$

轉完極坐標若錯誤(積分函數不是 r^2) 得 0 分. 轉對而範圍寫錯得 2 分

8. (15%) 求 $f(x,y)=x^2+xy+y^2$ 在橢圓上 $x^2+2xy+2y^2=1$ 之極值。

Solution:

Let $f(x,y)=x^2+xy+y^2$ and $g(x,y)=x^2+2xy+2y^2=1$

By Lagrange theorem,$\nabla f(x,y)=\lambda\nabla g(x,y)$[2pts]

$$\begin{cases} 2x+y & =\lambda(2x+2y) \quad \ldots(1) \\ x+2y & =\lambda(2x+4y) \quad \ldots(2) \\ x^2+2xy+2y^2-1 & =0 \quad \ldots(3) \end{cases} [3pts] \Rightarrow \begin{cases} \lambda=\frac{1}{2} & ,x=0 & ,y=\pm\frac{1}{\sqrt{2}} \\ \lambda=\frac{3}{2} & ,x=\pm\sqrt{2} & ,y=\mp\frac{1}{\sqrt{2}} \end{cases} [6pts]$$

The maximum of $f(x,y)$ is $f(\mp\sqrt{2},\pm\frac{1}{\sqrt{2}})=\frac{3}{2}$.[2pts]

The minimum of $f(x,y)$ is $f(0,\pm\frac{1}{\sqrt{2}})=\frac{1}{2}$. [2pts]