1022微乙01-05班期末考解答

1. (12%) 解微分方程 $y' = y^2(1-y)$, 滿足初值條件 $t = \ln 2$, $y = 2$.

Solution:

$$
\begin{aligned}
& y' = y^2(1-y) \\
\Rightarrow\ & \frac{y'}{y^2(1-y)} = 1 \\
\Rightarrow\ & \int \frac{y'dt}{y^2(1-y)} = \int dt \\
\Rightarrow\ & \int \frac{dy}{y^2(1-y)} = t + C && (3\, points) \\
\Rightarrow\ & \int \left(\frac{1}{y} + \frac{1}{y^2} + \frac{1}{1-y}\right) = t + C && (3\, points) \\
\Rightarrow\ & \ln|y| - \frac{1}{y} - \ln|1-y| = t + C && (3\, points)
\end{aligned}
$$

The solution satisfies $y(\ln 2) = 2$, so

$$
\begin{aligned}
\Rightarrow\ & \ln|2| - \frac{1}{2} - \ln|-1| = \ln 2 + C \\
\Rightarrow\ & C = -\frac{1}{2} && (3\, points)
\end{aligned}
$$

$$Ans : \ln\left|\frac{y}{1-y}\right| - \frac{1}{y} = t - \frac{1}{2}$$

2. (12%) 解微分方程 $y' + y = \sin t$.

Solution:

times integrating factor e^t (2%)

$$\begin{aligned} y' + y &= \sin t \\ \Rightarrow \quad e^t y' + e^t y &= e^t \sin t \\ \Rightarrow \quad (e^t y)' &= e^t \sin t \\ \Rightarrow \quad e^t y &= \int e^t \sin t dt \quad (2\%) \\ &= \frac{1}{2} e^t (\sin t - \cos t) + C, \quad C \text{ is a constant} \\ \Rightarrow \quad y &= \frac{1}{2}(\sin t - \cos t) + Ce^{-t}, \quad C \text{ is a constant} \quad (3\%) \end{aligned}$$

Moreover,

$$\begin{aligned} \int e^t \sin t dt &= -e^t \cos t + \int e^t \cos t dt \\ &= -e^t \cos t + e^t \sin t - \int e^t \sin t dt \\ \Rightarrow \quad \int e^t \sin t dt &= \frac{1}{2} e^t (\sin t - \cos t) \quad (5\%) \end{aligned}$$

3. (15%) 設服務員 A, B 的服務時間分別是隨機變數 X, Y, 都是指數分佈, 且互相獨立, 平均值分別是 $a, b > 0$.

(a) 寫出 X, Y 之機率密度函數,

(b) 寫出 X, Y 之聯合機率密度函數,

(c) 求 $P(X \geq Y)$.

Solution:

(a) [5pts]

$X \sim exponential(\lambda) \Rightarrow f_X(x) = \lambda e^{-\lambda x}$, for $x > 0 \Rightarrow E(X) = \frac{1}{\lambda}$ [2pts]

Therefore,

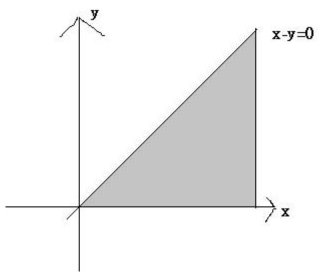
$$E[X] = a \Rightarrow \quad f_X(x) = \frac{1}{a} e^{\frac{-x}{a}}$$
$$E[Y] = b \Rightarrow \quad f_Y(y) = \frac{1}{b} e^{\frac{-y}{b}} \qquad [3pts]$$

(b) [5pts]

$f_{X,Y}(x, y) = f_X(x) \cdot f_Y(y)$ (Since X and Y be independent.) [2pts]

$f_{X,Y}(x, y) = \frac{1}{ab} e^{-\left(\frac{x}{a}+\frac{y}{b}\right)}$[3pts]

(c) [5pts]



$$Pr(X \geq Y) = Pr(X - Y \geq 0) = \int_0^{\infty} \int_y^{\infty} f_X(x) f_Y(y) dx\, dy \qquad [2pts]$$
$$= \frac{a}{a+b} \qquad [3pts]$$

4. (12%) 工安事故, 每 10 年有 5 件。設其遵守 Poisson 分佈。問 4 年無工安事故的機率為何？又 4 年內發生了 2 件工安事故的機率為何？

Solution:

Let λ be the average number of incidents per year

$$\lambda = \frac{5}{10} = \frac{1}{2}$$

and the time interval $T = 4$.
Since it follows the Poisson distribution, the probability of k incidents in the time interval T is

$$\frac{(\lambda T)^k}{k!} e^{-\lambda T} = \frac{2^k}{k!} e^{-2} \quad (8\%)$$

and thus

$$\begin{aligned} P(0 \text{ incidents in 4 year}) &= \frac{2^0}{0!} e^{-2} = e^{-2} \quad (2\%) \\ P(2 \text{ incidents in 4 year}) &= \frac{2^2}{2!} e^{-2} = 2e^{-2} \quad (2\%) \end{aligned}$$

5. (12%) 設 $f_X(t) = \lambda e^{-\lambda t}$, $t \geq 0$ 且 λ 為一大於 0 之常數. 求隨機變數 $\sqrt{X}$ 之機率密度函數.

Solution:

First find CDF of $Y = \sqrt{X}$

$F_Y(y) = P(Y \leq y) = P(0 \leq X \leq y^2) = \int_0^{y^2} \lambda e^{-\lambda t} dt$ (8 pt)

then $f_Y(y) = 2y\lambda e^{-\lambda y^2}$ for $y \geq 0$ (4 pt)

in fact , you may check it is a PDF

6. (12%) 某考試，應考者 1000 人，錄取 80 人。滿分為 100 分。已知平均 65 分，變異數 25 分^2。用柴比雪夫不等式估計，能確定錄取之最低分。

Solution:

We know that

$$P\left(|X - \mathrm{E}(X)| \geq k\sqrt{\mathrm{Var}(X)}\right) \leq \frac{1}{k^2} \quad \text{(4pts)}$$

and

$$\mathrm{E}(X) = 65 \quad \text{(2pts)}.$$

Note that

$$P\left(X - \mathrm{E}(X) \geq k\sqrt{\mathrm{Var}(X)}\right) \leq P\left(|X - \mathrm{E}(X)| \geq k\sqrt{\mathrm{Var}(X)}\right) \leq \frac{1}{k^2}.$$

If k statisfies $\frac{1}{k^2} \leq \frac{80}{1000}$, then $k = \frac{5}{\sqrt{2}}$ (4pts). Therefore,

$$P\left(X - 65 \geq \frac{5}{\sqrt{2}}\sqrt{25}\right) \leq \frac{1}{k^2} = \frac{8}{100}.$$

Hense, $X \geq 65 + \frac{25}{\sqrt{2}}$ (2pts).

7. (10%) 瑕積分 $\int_1^\infty \frac{dx}{x^p}$, 其中 $p > 0$, 對哪些 p 值會收斂？

Solution:
Case1: $p > 0$, $p \neq 1$:

$$\int_1^\infty \frac{1}{x^p} dx = \lim_{a \to \infty} \frac{a^{1-p}}{1-p} - \frac{1}{1-p} (5\text{pt})$$

hence when $p \neq 1$, $p > 1$ implies integral converges(4pt)
Case2 $p = 1$

$$\int_1^\infty \frac{1}{x} dx = \lim_{a \to \infty} lna = \infty (1\text{pt})$$

8. (15%) 隨機變數 X, Y 分別取值 $\{1,2\}$ 及 $\{1,2,3\}$.

$$\text{設 } P(X=1,Y=1)=\frac{2}{7},\ P(X=1,Y=2)=\frac{3}{14},\ P(X=1,Y=3)=\frac{1}{7}$$
$$P(X=2,Y=1)=\frac{3}{14},\ P(X=2,Y=2)=\frac{1}{14}, P(X=2,Y=3)=\frac{1}{14}$$

(a) (5%) 求 $\mathrm{E}(X)$, $\mathrm{E}(Y)$,

(b) (5%) 求 $\mathrm{Var}(X)$, $\mathrm{Var}(Y)$,

(c) (5%) 問 X, Y 是否獨立？

Solution:

$P(X=1) \ = \dfrac{9}{14}$, $P(X=2) = \dfrac{5}{14}$

$P(Y=1) \ = \dfrac{1}{2}$, $P(Y=2) = \dfrac{2}{7}$, $P(Y=3) = \dfrac{3}{14}$

(a)

$E(X) = 1\cdot\dfrac{9}{14} + 2\cdot\dfrac{5}{14} = \dfrac{19}{14} \quad (2pts)$

$E(Y) = 1\cdot\dfrac{1}{2} + 2\cdot\dfrac{2}{7} + 3\cdot\dfrac{3}{14} = \dfrac{12}{7} \quad (3pts)$

(b)

$Var(X) = E(X^2) - (E(X))^2 = \dfrac{29}{14} - (\dfrac{19}{14})^2 = \dfrac{45}{196} \quad (2pts)$

$Var(Y) = E(Y^2) - (E(Y))^2 = \dfrac{25}{7} - (\dfrac{12}{7})^2 = \dfrac{31}{49} \quad (3pts)$

(c)

$P(X=1,Y=1) = \dfrac{2}{7} \quad (1pts)$

$P(X=1)P(Y=1) = \dfrac{9}{28} \quad (1pts)$

$P(X=1,Y=1) \neq P(X=1)P(Y=1) \quad (3pts)$

Hence, X and Y are NOT independent.

If you are not explain "Why X and Y are not independent?"
You can't get any credit.