1022微甲01-04班期末考解答

1. (10%) Evaluate $\int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1}x}{x}\,dx$. [Hint. Express it as an iterated integral.]

Solution:

(Method I)

Consider $f(x,y) = \frac{tan^{-1}yx}{x} \Rightarrow \frac{\partial}{\partial y}f(x,y) = \frac{1}{1+(xy)^2}$ (1 %)

$\Rightarrow \int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1}x}{x}\,dx = \int_0^\infty \int_1^\pi \frac{1}{1+(xy)^2}\,dydx$ (2 %)

Then by Fubini's Thm.(2 %)

$\Rightarrow \int_0^\infty \int_1^\pi \frac{1}{1+(xy)^2}\,dydx = \int_1^\pi \int_0^\infty \frac{1}{1+(xy)^2}\,dxdy$ (1 %)

$$\Rightarrow \int_1^\pi \int_0^\infty \frac{1}{1+(xy)^2}\,dxdy = \int_1^\pi \frac{tan^{-1}yx}{y}\Big|_0^\infty\,dy = \int_1^\pi \frac{1}{y}\cdot\frac{\pi}{2}\,dy(2\ \%)$$
$$= \frac{\pi}{2}\ln y|_1^\pi = \frac{\pi}{2}\ln\pi(2\ \%)$$

(Method II)

$\int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1}x}{x}\,dx = \int_0^\infty \int_x^{\pi x} \frac{1}{x}\cdot\frac{1}{1+y^2}\,dydx(2\ \%)$

Then by Fubini's Thm.(2 %)

$\Rightarrow \int_0^\infty \int_x^{\pi x} \frac{1}{x}\cdot\frac{1}{1+y^2}\,dydx = \int_0^\infty \int_{\frac{y}{\pi}}^{y} \frac{1}{x}\cdot\frac{1}{1+y^2}\,dxdy(3\ \%)$

$\Rightarrow \int_0^\infty \int_{\frac{y}{\pi}}^{y} \frac{1}{x}\cdot\frac{1}{1+y^2}\,dxdy = \int_0^\infty \ln x|_{\frac{y}{\pi}}^{y}\cdot\frac{1}{1+y^2}\,dy = \ln\pi\cdot\int_0^\infty \frac{1}{1+y^2}\,dy = \frac{\pi}{2}\ln\pi(3\ \%)$

(Method III)

$\int_0^\infty \frac{\tan^{-1}\pi x - \tan^{-1}x}{x}\,dx = \int_0^\infty \int_{\tan^{-1}x}^{\tan^{-1}\pi x} \frac{1}{x}\,dydx(2\ \%)$

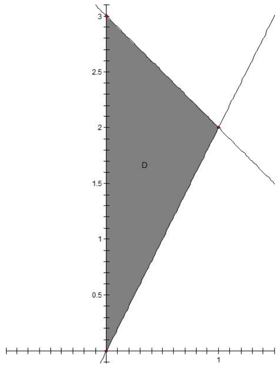
Then by Fubini's Thm.(2 %)

$\Rightarrow \int_0^\infty \int_{\tan^{-1}x}^{\tan^{-1}\pi x} \frac{1}{x}\,dydx = \int_0^{\frac{\pi}{2}} \int_{\frac{\tan y}{\pi}}^{\tan y} \frac{1}{x}\,dxdy(3\ \%)$

$\Rightarrow \int_0^{\frac{\pi}{2}} \int_{\frac{\tan y}{\pi}}^{\tan y} \frac{1}{x}\,dxdy = \int_0^{\frac{\pi}{2}} \ln x|_{\frac{\tan y}{\pi}}^{\tan y}dy = \frac{\pi}{2}\ln\pi(3\ \%)$

2. (10%) Evaluate $\displaystyle\int_{\tan^{-1} 2}^{\frac{\pi}{2}} \int_0^{\frac{3}{\cos\theta+\sin\theta}} r^3 \cos\theta \sin\theta \, dr d\theta.$

Solution:



(Method I)

$r = \dfrac{3}{\cos\theta + \sin\theta} \Rightarrow r\cos\theta + r\sin\theta = 3 \Rightarrow x + y = 3$ (2 %)

$\Rightarrow \displaystyle\int_{\tan^{-1} 2}^{\frac{\pi}{2}} \int_0^{\frac{3}{\cos\theta+\sin\theta}} r^3 \cos\theta \sin\theta \; dr d\theta = \int\int_D xy \; dA$ (3 %)

Then by Fubini's Thm.

$$\begin{aligned}\int\int_D xy \; dA &= \int_0^1 \int_{2x}^{3-x} xy \; dydx \text{ (2 \%)} \\ &= \int_0^1 x \cdot \frac{1}{2}[(3-x)^2 - (2x)^2] \; dx \\ &= \frac{1}{2}\int_0^1 9x - 6x^2 - 3x^3 \; dx = \frac{7}{8} \text{ (3 \%)}\end{aligned}$$

(Method II)

$\displaystyle\int_{\tan^{-1} 2}^{\frac{\pi}{2}} \int_0^{\frac{3}{\cos\theta+\sin\theta}} r^3 \cos\theta \sin\theta \; dr d\theta = \int_{\tan^{-1} 2}^{\frac{\pi}{2}} \frac{1}{4}\left(\frac{3}{\cos\theta+\sin\theta}\right)^4 \cos\theta\sin\theta \; d\theta$

(3 %)

$\Rightarrow \displaystyle\frac{81}{4}\int_{\tan^{-1} 2}^{\frac{\pi}{2}} \frac{\cos\theta\sin\theta}{(\cos\theta+\sin\theta)^4} \; d\theta = \frac{81}{4}\int_{\tan^{-1} 2}^{\frac{\pi}{2}} \frac{\sec^2\theta\tan\theta}{(1+\tan\theta)^4} \; d\theta$(1 %)

Let $u = 1 + \tan\theta \Rightarrow du = \sec^2\theta d\theta$(2 %)

$$\begin{aligned}\Rightarrow \frac{81}{4}\int_{\tan^{-1} 2}^{\frac{\pi}{2}} \frac{\sec^2\theta\tan\theta}{(1+\tan\theta)^4} \; d\theta &= \frac{81}{4}\int_3^{\infty} \frac{u-1}{u^4} \; du = \frac{81}{4} \cdot \left(\frac{-1}{2}u^{-2} - \frac{-1}{3}u^{-3}\right)_3^{\infty} \\ &= \frac{81}{4}\left(\frac{1}{18} - \frac{1}{81}\right) = \frac{7}{8} \text{ (4 \%)}\end{aligned}$$

3. (10%) Evaluate $\iiint\limits_{B} (x^2+y^2+z^2)^2 \, dV$, where B is the ball with center the origin and radius 1.

Solution:
我們令

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

則我們有

$$\begin{aligned} Ans &= \iiint_B (x^2+y^2+z^2)^2 \, dV \\ &= \int_0^{2\pi} \int_0^{\pi} \int_0^{1} \left(\rho^2\right)^2 \cdot \rho^2 \sin\phi \; d\rho \, d\phi \, d\theta \\ &= \int_0^{2\pi} d\theta \cdot \int_0^{\pi} \sin\phi \, d\phi \cdot \int_0^{1} \rho^6 \, d\rho \\ &= \theta\Big|_0^{2\pi} \cdot (-\cos\phi)\Big|_0^{\pi} \cdot \frac{1}{7}\rho^7\Big|_0^{1} \\ &= 2\pi \cdot 2 \cdot \frac{1}{7} \\ &= \frac{4}{7}\pi \end{aligned}$$

4. (10%) Let E be the tetrahedron bounded by planes $-x+y+z=0$, $x-y+z=0$, $x+y-z=0$, and $-x+5y+7z=6$.

(a) Find the volume of E.

(b) Evaluate $\displaystyle\iiint_E z\,dV$.

Solution:

我們令

$$\begin{cases} u=-x+y+z \\ v=x-y+z \\ w=x+y-z \end{cases}$$

反解回去，可以得到

$$\begin{cases} x=\dfrac{1}{2}(v+w) \\ \\ y=\dfrac{1}{2}(u+w) \\ \\ z=\dfrac{1}{2}(u+v) \end{cases}$$

故

$$\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial w} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \dfrac{\partial y}{\partial w} \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \dfrac{\partial z}{\partial w} \end{vmatrix}=\begin{vmatrix} 0 & \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2} & 0 & \dfrac{1}{2} \\ \dfrac{1}{2} & \dfrac{1}{2} & 0 \end{vmatrix}=\frac{1}{4}$$

另一方面，$-x+5y+7z=6$ 經過變換之後，變成 $u+\frac{1}{2}v+\frac{1}{3}w=1$。所以$u$、$v$、$w$的範圍為

$$\begin{cases} u=0 \\ v=0 \\ w=0 \\ u+\dfrac{1}{2}v+\dfrac{1}{3}w=1 \end{cases}$$

這是一個放的好好的直角四面體，其體積 $volume(E')$ 為 $\frac{1}{6}\times 1\times 2\times 3=1$。

有了以上的計算，我們就可以算 (a) 小題了！

$$\begin{aligned} volume(E) &= \iiint_E dV=\iiint_E dx\,dy\,dz \\ &= \iiint_{E'}\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \\ &= \frac{1}{4}\iiint_{E'} du\,dv\,dw \\ &= \frac{1}{4}\cdot volume(E') \\ &= \frac{1}{4}\times 1=\frac{1}{4} \end{aligned}$$

若是高中數學還記得的話，(a) 也可以用高中的辦法來做。不難發現，原本四面體的四個頂點分別為 $(0,0,0)$、$(0,\frac{1}{2},\frac{1}{2})$、$(1,0,1)$、$(\frac{3}{2},\frac{3}{2},0)$，故四面體的體積為

$$volume(E)=\frac{1}{6}|\begin{vmatrix} 0 & \dfrac{1}{2} & \dfrac{1}{2} \\ 1 & 0 & 1 \\ \dfrac{3}{2} & \dfrac{3}{2} & 0 \end{vmatrix}|=\frac{1}{4}\times 1=\frac{1}{4}$$

當然，(a) 小題也可以硬算，計算過程就不贅述了。
至於(b)，沒什麼辦法的話就硬算吧！不過硬算之前，再做一次變數變換會稍微好算一點。令

$$\begin{cases} \alpha = u \\ \beta = \dfrac{1}{2}v \\ \gamma = \dfrac{1}{3}w \end{cases}$$

所以 $u + \frac{1}{2}v + \frac{1}{3}w = 1$ 變成 $\alpha + \beta + \gamma = 1$，而$\frac{\partial(u,v,w)}{\partial(\alpha,\beta,\gamma)} = 6$。
我們得到

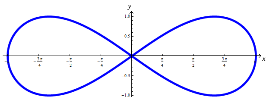
$$\begin{aligned}
\iiint_E z\,dV &= \iiint_{E'} \frac{1}{2}(u+v)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw \\
&= \iiint_{E''} \left(\frac{1}{2}\alpha + \beta\right)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\left|\frac{\partial(u,v,w)}{\partial(\alpha,\beta,\gamma)}\right| d\alpha\,d\beta\,d\gamma \\
&= \frac{1}{4}\times 6\int_0^1\int_0^{1-\gamma}\int_0^{1-\beta-\gamma}\left(\frac{1}{2}\alpha+\beta\right) d\alpha\,d\beta\,d\gamma \\
&= \frac{3}{2}\int_0^1\int_0^{1-\gamma}\left(\frac{1}{4}\alpha^2+\beta\alpha\right)\Big|_0^{1-\beta-\gamma} d\beta\,d\gamma \\
&= \frac{3}{8}\int_0^1\int_0^{1-\gamma}(1+2\beta-3\beta^2-2\beta\gamma-2\gamma+\gamma^2)\,d\beta\,d\gamma \\
&= \frac{3}{8}\int_0^1 [\beta^2-\beta^3-\beta^2\gamma+(\gamma-1)^2\beta]\Big|_0^{1-\beta} d\gamma \\
&= \frac{3}{8}\int_0^1 (1-\gamma)^3\,d\gamma \\
&= \frac{3}{8}\left[-\frac{1}{4}(1-\gamma)^4\right]\Big|_0^1 \\
&= \frac{3}{32}
\end{aligned}$$

不過，若是熟悉四面體重心的位置的話，這一題其實可以速解：

$$\begin{aligned}
\iiint_E z\,dV &= \bar{z}\cdot volume(E) \\
&= \frac{1}{2}(\bar{u}+\bar{v})\cdot\frac{1}{4} \\
&= \frac{1}{2}\left(\frac{1}{4}+\frac{1}{2}\right)\cdot\frac{1}{4} = \frac{3}{32}
\end{aligned}$$

其中，$\bar{u} = \frac{1}{4}(0+1+0+0) = \frac{1}{4}$、$\bar{v} = \frac{1}{4}(0+0+2+0) = \frac{1}{2}$。

5. (10%) Let C be the upper half of the curve $(x^2+y^2)^2 = x^2 - y^2$. Evaluate $\int_C y\,ds$.



Solution:

In polar coordinates, with $x = r\cos\theta$ and $y = r\sin\theta$, the curve becomes $r^4 = r^2(\cos^2\theta - \sin^2\theta) \Rightarrow r^2 = \cos 2\theta$. (3%)

By symmetry,

$$\int_C y\,ds = 2\int_{C_1} y\,ds$$

where C_1 is the part in the first quadrant and corresponds to the curve $r(\theta) = \sqrt{\cos 2\theta}$, $\theta \in [0, \frac{\pi}{4}]$. (1% for parameter range)

$$\begin{aligned}
r'(\theta) =& \frac{-\sin 2\theta}{\sqrt{\cos 2\theta}} \\
\int_C y\,ds =& 2\int_{C_1} y\,ds \\
=& 2\int_{C_1} y(\theta)\sqrt{(r(\theta))^2 + (r'(\theta))^2}d\theta && (1\%) \\
=& 2\int_0^{\frac{\pi}{4}} \sqrt{\cos 2\theta}\,\sin\theta\sqrt{\cos 2\theta + \frac{\sin^2 2\theta}{\cos 2\theta}}\,d\theta && (2\%) \\
=& 2\int_0^{\frac{\pi}{4}} \sqrt{\cos 2\theta}\,\sin\theta\,\frac{1}{\sqrt{\cos 2\theta}}\,d\theta \\
=& 2\int_0^{\frac{\pi}{4}} \sin\theta\,d\theta \\
=& 2 - \sqrt{2} && (3\%)
\end{aligned}$$

(Note that $ds = \sqrt{(r(\theta))^2 + (r'(\theta))^2}\,d\theta$ can also be obtained by $ds = \sqrt{(\frac{dx(\theta)}{d\theta})^2 + (\frac{dy(\theta)}{d\theta})^2}\,d\theta$.)

6. (10%) Evaluate $\int_C (yze^{xyz}+x)\,dx+xze^{xyz}\,dy+xye^{xyz}\,dz$, where C is the curve $\mathbf{r}(t)=\langle t,\,\cos(\pi t),\,\tan^{-1}t\rangle$, $0\le t\le 1$.

Solution:

The fact that $\mathbf{F}$ is conservative can be shown by either (1) calculating $\nabla\times\mathbf{F}$ and finding that it is $\mathbf{0}$, or (2) finding a scalar function f and show that $\mathbf{F}=\nabla f$.
The function f can be found as follows:

$$\frac{\partial f}{\partial x}=yxe^{xyz}+x$$
$$\Rightarrow f=e^{xyz}+\frac{1}{2}x^2+h(y,z)$$

$$\frac{\partial f}{\partial y}=xze^{xyz}+\frac{\partial}{\partial y}h(y,z)\triangleq xze^{xyz}$$
$$\Rightarrow h(y,z)=g(z),f=e^{xyz}+\frac{1}{2}x^2+g(z)$$

$$\frac{\partial f}{\partial z}=xye^{xyz}+\frac{d}{dz}g(z)\triangleq xye^{xyz}$$
$\Rightarrow g(z)=C$ and we can take $C=0$.
$$\Rightarrow h(y,z)=g(z),f=e^{xyz}+\frac{1}{2}x^2 \qquad (6\%)$$

By the Fundamental Theorem for Line Integrals, (2%)

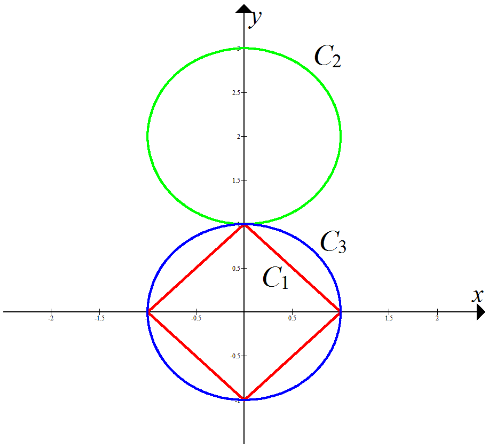
$$\begin{aligned}&\int_C (yze^{xyz}+x)\,dx+(xze^{xyz})\,dy+(xye^{xyz})\,dz\\
=&\int_C \nabla f\cdot d\mathbf{r}\\
=&f(\mathbf{r}(1))-f(\mathbf{r}(0))\\
=&f(1,-1,\frac{\pi}{4})-f(0,1,0)\\
=&e^{-\frac{\pi}{4}}-\frac{1}{2} \qquad (2\%)\end{aligned}$$

Note that the integral can also be evaluated as

$$\begin{aligned}&\int_C (\nabla e^{xyz})\cdot d\mathbf{r}+\int_C x\,dx\\
=&[e^{xyz}]_{(0,1,0)}^{(1,-1,\frac{\pi}{4})}+\int_0^1 x(t)\cdot x'(t)dt\\
=&(e^{-\frac{\pi}{4}}-1)+\int_0^1 t\,dt\\
=&e^{-\frac{\pi}{4}}-\frac{1}{2}\end{aligned}$$

7. (10%) Let the vector field $\mathbf{F}(x,y) = \dfrac{x^2y}{(x^2+y^2)^2}\,\mathbf{i} - \dfrac{x^3}{(x^2+y^2)^2}\,\mathbf{j}$, C_1 be the curve $|x|+|y|=1$ and C_2 be the curve $x^2+(y-2)^2=1$. Find $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r}$ and $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r}$.

Solution:



We first notice that $(0,0)$ is not in the domain of $\mathbf{F}$.
Let

$$P(x,y) = \frac{x^2y}{(x^2+y^2)^2} \text{ and } Q(x,y) = -\frac{x^3}{(x^2+y^2)^2}.$$

Step 1: $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.

(3 points. One gets only 1 point if just writing $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$.)

We have

$$\frac{\partial P}{\partial y} = \frac{x^2(x^2+y^2)^2 - 4x^2y^2(x^2+y^2)}{(x^2+y^2)^4} = \frac{x^4-3x^2y^2}{(x^2+y^2)^3},$$
$$\frac{\partial Q}{\partial x} = \frac{-3x^2(x^2+y^2)^2 + 4x^4(x^2+y^2)}{(x^2+y^2)^4} = \frac{x^4-3x^2y^2}{(x^2+y^2)^3}.$$

Step 2: $\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = -\pi$.

Let C_3 be the unit circle with positive orientation (i.e. counter clockwise).
Let D_1 be the region between C_1 and C_3.
(2 points) By Green's Theorem, we have

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_3} \mathbf{F}\cdot d\mathbf{r} + \iint_{D_1} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}dA.$$

(2 points) By Step 1, we have $\dfrac{\partial Q}{\partial x} - \dfrac{\partial P}{\partial y} = 0.$, Therefore,

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \left(\frac{x^2y}{(x^2+y^2)^2}, -\frac{x^3}{(x^2+y^2)^2}\right)\cdot d(x,y)$$

$$= \int_0^{2\pi} (\cos^2 t\sin t, -\cos^3 t)\cdot(-\sin t, \cos t)dt = -\int_0^{2\pi} \cos^2 t dt = -4\times\frac{1}{2}\times\frac{\pi}{2} = -\pi.$$

(One distinguished the orientation incorrectly and got π. $\Rightarrow$ 1 point.)

Step 3 (3 points): $\oint_{C_2} \mathbf{F}\cdot d\mathbf{r} = 0$.

Let D_2 be the simply connected region enclosed by C_2.

Method 1

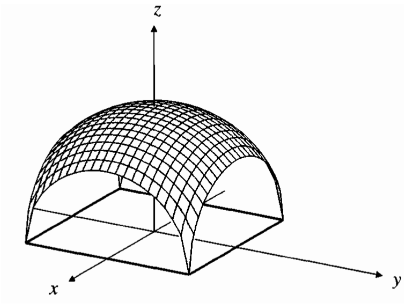
Since $\mathbf{F}$ is conservative by Step 1, it is path independent.
We have $\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0$.

Method 2
One can verify that $P(x,y)$ and $Q(x,y)$ have continuous partial derivatives.
By Green's Theorem, we have

$$\oint_{C_2} \mathbf{F} \cdot d\mathbf{r} = \iint_{D_2} \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} dA = 0.$$

8. (15%) (a) Find the area of the part of the sphere $x^2+y^2+z^2=2$ that lies above the plane $z=1$.

(b) Let the canopy be the part of the upper hemisphere $x^2+y^2+z^2=2$ that lies above the square $-1\le x\le 1$, $-1\le y\le 1$, and let C be the boundary of canopy oriented counterclockwise when viewed from above. Evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$, where $\mathbf{F}(x,y,z)=(xz+\tan x^2)\,\mathbf{i}+(\sin x\cos y+e^{y^2})\,\mathbf{j}+\left(-\frac{y^2}{2}+\sin\sqrt{z}\right)\mathbf{k}$.



Solution:

(a)作法一

利用原來的x,y參數化$x^2+y^2+z^2=2$的上半部分，得到$z=\sqrt{2-x^2-y^2}$。接著計算$\frac{\partial z}{\partial x}$和$\frac{\partial z}{\partial x}$

$$\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{2-x^2-y^2}},\frac{\partial z}{\partial y}=\frac{-y}{\sqrt{2-x^2-y^2}}$$

最後代入曲面表面積的定義:

$$A(S)=\int\int_D\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}dxdy=\int\int_{x^2+y^2=1}\frac{\sqrt{2}}{\sqrt{2-x^2-y^2}}dxdy=\sqrt{2}\int_0^{2\pi}\int_0^1\frac{1}{\sqrt{2-r^2}}rdrd\theta=(4-2\sqrt{2})\pi$$

(倒數第二個等號是利用極坐標代換，另$x=r\cos\theta$, $y=r\sin\theta$) (a)作法二

利用球坐標(spherical coordinate)參數化$x^2+y^2+z^2=2$的上半部分，得到

$$x=\sqrt{2}\cos\theta\sin\phi,y=\sqrt{2}\sin\theta\sin\phi,z=\sqrt{2}\cos\phi$$

其中$0\le\theta\le 2\pi$、$0\le\phi\le\frac{\pi}{4}$($z=1$的上方)。 令$r(\theta,\phi)=(\sqrt{2}\cos\theta\sin\phi,\sqrt{2}\sin\theta\sin\phi,\sqrt{2}\cos\phi)$，接著計算$\frac{\partial r}{\partial\theta}$和$\frac{\partial r}{\partial\phi}$

$$\frac{\partial r}{\partial\theta}=(-\sqrt{2}\sin\theta\sin\phi,\sqrt{2}\cos\theta\sin\phi,0),\frac{\partial r}{\partial\phi}=(\sqrt{2}\cos\theta\cos\phi,\sqrt{2}\sin\theta\cos\phi,-\sqrt{2}\sin\phi)$$

計算$\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\phi}$和$|\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\phi}|$

$$\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\phi}=(-2\cos\theta\sin^2\phi,-2\sin\theta\sin^2\phi,-2\sin\phi\cos\phi)$$

$$|\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\phi}|=\sqrt{(-2\cos\theta\sin^2\phi)^2+(-2\sin\theta\sin^2\phi)^2+(-2\sin\phi\cos\phi)^2}=2|\sin\phi|$$

因此所求的曲面表面積為

$$A(S)=\int\int_D|\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial\phi}|d\theta d\phi=\int_0^{\frac{\pi}{4}}\int_0^{2\pi}2\sin\phi d\theta d\phi=2(\int_0^{\frac{\pi}{4}}\sin\phi d\phi)(\int_0^{2\pi}d\theta)=(4-2\sqrt{2})\pi$$

(a)作法三

令$S(r,\theta)=(r\cos\theta,r\sin\theta,\sqrt{2-r^2})$，其中$0\le\theta\le 2\pi$、$0\le r\le 1$，注意到它是$x^2+y^2+z^2=2$在$z=1$上方的一種參數化。接著計算$\frac{\partial S}{\partial\theta}$和$\frac{\partial S}{\partial r}$

$$\frac{\partial S}{\partial\theta}=(\cos\theta,\sin\theta,\frac{-r}{\sqrt{2-r^2}}),\frac{\partial S}{\partial r}=(-r\sin\theta,r\cos\theta,0)$$

計算$\frac{\partial S}{\partial r}\times\frac{\partial S}{\partial\theta}$和$|\frac{\partial S}{\partial r}\times\frac{\partial S}{\partial\theta}|$

$$\frac{\partial S}{\partial r}\times\frac{\partial S}{\partial\theta}=(\frac{r^2\cos\theta}{\sqrt{2-r^2}},\frac{-r^2\sin\theta}{\sqrt{2-r^2}},r),|\frac{\partial S}{\partial r}\times\frac{\partial S}{\partial\theta}|=\frac{\sqrt{2}r}{\sqrt{2-r^2}}$$

因此所求的曲面表面積為

$$A(S)=\int\int_D|\frac{\partial S}{\partial r}\times\frac{\partial S}{\partial\theta}|drd\theta=\sqrt{2}\int_0^{2\pi}\int_0^1\frac{1}{\sqrt{2-r^2}}rdrd\theta=(4-2\sqrt{2})\pi$$

(a)作法四

令$r(\theta,z)=(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta,z)$，其中$0\leq\theta\leq 2\pi$、$1\leq z\leq\sqrt{2}$，注意到它是$x^2+y^2+z^2=2$在$z=1$上方的一種參數化。接著計算$\frac{\partial r}{\partial\theta}$和$\frac{\partial r}{\partial z}$

$$\frac{\partial r}{\partial\theta}=(-\sqrt{2}\sin\theta,\sqrt{2}\cos\theta,0),\frac{\partial r}{\partial z}=(0,0,1)$$

計算$\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial z}$和$|\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial z}|$

$$\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial z}=(\sqrt{2}\cos\theta,\sqrt{2}\sin\theta,0),|\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial z}|=\sqrt{(\sqrt{2}\cos\theta)^2+(\sqrt{2}\sin\theta)^2+(0)^2}=\sqrt{2}$$

因此所求的曲面表面積為

$$A(S)=\int\int_D|\frac{\partial r}{\partial\theta}\times\frac{\partial r}{\partial z}|d\theta dz=\int_0^{2\pi}\int_1^{\sqrt{2}}\sqrt{2}dzd\theta=(4-2\sqrt{2})\pi$$

(a)作法五

令$r=\sqrt{x^2+y^2}$，因此$r=\sqrt{2-z^2}$。接著利用加總圓周的方式來計算曲面表面積:

$$ds=\sqrt{(dz)^2+(dr)^2}=\sqrt{1+(\frac{dr}{dz})^2}dz=\sqrt{1+(\frac{-z}{\sqrt{2-z^2}})^2}dz$$

$$A(S)=\int 2\pi rds=\int_1^{\sqrt{2}}2\pi\sqrt{2-z^2}\sqrt{1+(\frac{-z}{\sqrt{2-z^2}})^2}dz=2\pi\int_1^{\sqrt{2}}\sqrt{2}dz=(4-2\sqrt{2})\pi$$

(b)作法一

注意到這個帳棚(canopy)是一個有piecewise smooth boundary的曲面，所以我們可以利用Stoke's Theorem 將該線積分轉換成面積分:

$$\oint_C\mathbf{F}\cdot d\mathbf{r}=\int\int_S(\nabla\times\mathbf{F})\cdot d\mathbf{S}=\int\int_S(\nabla\times\mathbf{F})\cdot\mathbf{n}dS$$

首先先參數化該帳篷，令$S(x,y)=(x,y,\sqrt{2-x^2-y^2})$，接著計算$\nabla\times\mathbf{F}$和$\mathbf{n}$

$$\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i}&\mathbf{j}&\mathbf{k}\\ \frac{\partial}{\partial x}&\frac{\partial}{\partial y}&\frac{\partial}{\partial z}\\ xz+\tan(x^2)&\sin(x)\cos(y)+e^{y^2}&-\frac{y^2}{2}+\sin(\sqrt{z})\end{vmatrix}=-y\mathbf{i}+x\mathbf{j}+\cos(x)\cos(y)\mathbf{k}$$

$$\frac{\partial S}{\partial x}=(1,0,\frac{-x}{\sqrt{2-x^2-y^2}}),\frac{\partial S}{\partial y}=(1,0,\frac{-y}{\sqrt{2-x^2-y^2}}),\frac{\partial S}{\partial x}\times\frac{\partial S}{\partial y}=(\frac{x}{\sqrt{2-x^2-y^2}},\frac{y}{\sqrt{2-x^2-y^2}},1)$$

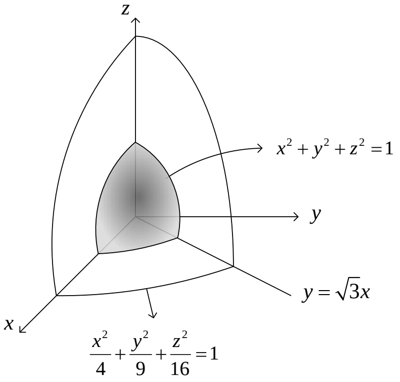
所以，所求線積分

$$\begin{aligned}\int\int_S(\nabla\times\mathbf{F})\cdot\mathbf{n}dS&=\int\int_D(\nabla\times\mathbf{F})\cdot\frac{\partial S}{\partial x}\times\frac{\partial S}{\partial y}dA\\&=\int_{-1}^1\int_{-1}^1(-y,x,\cos x\cos y)\cdot(\frac{x}{\sqrt{2-x^2-y^2}},\frac{y}{\sqrt{2-x^2-y^2}},1)dxdy=4\sin^2(1)\end{aligned}$$

9. (15%) Let $\mathbf{F}(x,y,z) = \dfrac{1}{(x^2+y^2+z^2)^{3/2}}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$.

(a) Let S_1 be the part of the sphere $x^2+y^2+z^2=1$ in the first octant bounded by the planes $y=0$, $y=\sqrt{3}x$ and $z=0$, oriented upward. Find the flux of $\mathbf{F}$ across S_1.

(b) Let S_2 be the part of the surface $\dfrac{x^2}{4}+\dfrac{y^2}{9}+\dfrac{z^2}{16}=1$ in the first octant bounded by the planes $y=0$, $y=\sqrt{3}x$ and $z=0$, oriented upward. Find the flux of $\mathbf{F}$ across S_2.



Solution:

(a)

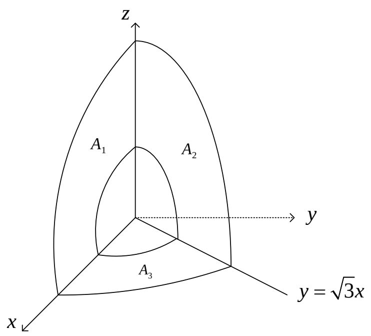
$$\int_{S_1} \mathbf{F}\cdot d\mathbf{s} = \int_{S_1} \mathbf{F}\cdot\mathbf{n}ds = \int_{S_1} \frac{(x,y,z)}{R^3}\cdot\frac{(x,y,z)}{R}ds = \int_{S_1}\frac{1}{R^2}ds \xlongequal{R=1 \text{ on } S_1} \int_{S_1} ds = 4\pi\cdot 1^2\cdot\frac{1}{2}\cdot\frac{1}{6} = \frac{\pi}{3} \quad (6\ 分)$$

(b)

$$\begin{aligned}\nabla\cdot\mathbf{F} &= \frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2+z^2}\right)+\frac{\partial}{\partial z}\left(\frac{z}{x^2+y^2+z^2}\right)\\ &= \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3x^2}{(x^2+y^2+z^2)^{\frac{53}{2}}} + \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3y^2}{(x^2+y^2+z^2)^{\frac{53}{2}}}\\ &\quad + \frac{1}{(x^2+y^2+z^2)^{\frac{3}{2}}} - \frac{3z^2}{(x^2+y^2+z^2)^{\frac{53}{2}}}\\ &= \frac{3}{(x^2+y^2+z^2)^{\frac{3}{2}}} - 3\frac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{\frac{53}{2}}} = 0\end{aligned}$$

$\nabla\cdot\mathbf{F} = 0$ on $\mathbb{R}^3\setminus\{(0,0,0)\}$ (4 分)

Let E be the solid bdd by $\begin{cases} S_1 \\ S_2 \\ y=0 \\ z=0 \\ y=\sqrt{3}x \end{cases}$ $\because$ $\mathbf{F} \,/\!/\, (x,y,z)$ and outward normals $\mathbf{n}$ of A_1, A_2, A_3 are $\perp$ (x,y,z)



$$\therefore \int_{A_1+A_2+A_3} \mathbf{F}\cdot\mathbf{n}ds = 0 \quad (3\ 分)$$

$$\int_E \nabla\cdot\mathbf{F}dV = \int_{S_1+S_2+A_1+A_2+A_3}\mathbf{F}\cdot\mathbf{n}ds = \int_{S_1}\mathbf{F}\cdot\underbrace{\mathbf{n}}_{\text{inward with respect to part (a)}}ds + \int_{S_2}\mathbf{F}\cdot\mathbf{n}ds = -\frac{\pi}{3}+\int_{S_2}\mathbf{F}\cdot\mathbf{n}ds$$

$\because$ LHS= 0 $\therefore \int_{S_2} \mathbf{F} \cdot \mathbf{n} ds = \dfrac{\pi}{3}$ (2 分)

10. (10%) Solve the differential equation $y'' + 2y' + y = x^{-3}e^{-x}$ with initial conditions $y(1) = y'(1) = 0$. Find $\lim_{x\to\infty} y(x)$.

Solution:

Criteria

Step 1. Find the solution of the complementary equation. (2 points.)
Step 2. Find the particular solution. (4 points.)
Step 3. Find the solution with the initial conditions. (2 points.)
Step 4. Find the limit. (2 points.)

Solution

Step 1. The complementary equation is $y'' + 2y' + y = 0$.
The root of auxiliary equation $r^2 + 2r + 1 = 0$ is $r = -1$, repeated.
So the general solution of complementary equation is (2 points)

$$y_c = c_1e^{-x} + c_2xe^{-x}$$

for constants c_1, c_2.

Step 2. We want to find the particular solution.

Method 1: Use variation of parameter.
The particular solution is of the form (1 point)

$$y_p = u_1(x)e^{-x} + u_2(x)xe^{-x}.$$

One has

$$\begin{cases} u_1'e^{-x} + u_2'xe^{-x} = 0 \\ u_1'(e^{-x})' + u_2'(xe^{-x})' = x^{-3}e^{-x}, \end{cases}$$

or equivalently (1 point),

$$\begin{cases} u_1' + u_2'x = 0 \\ -u_1' + u_2'(1-x) = x^{-3}. \end{cases}$$

So we find $\begin{cases} u_1' = -x^{-2} \\ u_2' = x^{-3}. \end{cases}$ and $\begin{cases} u_1 = x^{-1} \\ u_2 = -\frac{1}{2}x^{-2} \end{cases}$ (1 point).
The particular solution is (1 point)

$$y_p = x^{-1}e^{-x} + -\frac{1}{2}x^{-2}xe^{-x} = \frac{1}{2}x^{-1}e^{-x}.$$

Method 2.

$$y'' + 2y' + y = x^{-3}e^{-x} \Leftrightarrow e^xy'' + 2e^xy' + e^xy = x^{-3}$$
$$\Leftrightarrow (e^xy)'' = x^{-3} \Leftrightarrow (e^xy)' = -\frac{1}{2}x^{-2} + c_2$$
$$\Leftrightarrow e^xy = \frac{1}{2}x^{-1} + c_1 + c_2x \Leftrightarrow y = (c_1 + c_2x + \frac{1}{2}x^{-1})e^{-x}$$

for some constants c_1 and c_2.

So we find the general solution is

$$y = (c_1 + c_2x + \frac{1}{2}x^{-1})e^{-x}.$$

Step 3. With the initial conditions $y(1) = y'(1) = 0$, we have

$$\begin{cases} y(1) = (c_1 + c_2 + \frac{1}{2})e^{-1} = 0 \\ y'(1) = (c_2 - \frac{1}{2})e^{-1} - (c_1 + c_2 + \frac{1}{2})e^{-1} = 0, \end{cases}$$

that is,

$$\begin{cases} y(1) = c_1 + c_2 = -\frac{1}{2} \\ y'(1) = -c_1 - 1 = 0. \end{cases}$$

We find $c_1 = -1$ and $c_2 = \frac{1}{2}$. (2 points. One lost 1 point while writing $c_1 = 1$.)
So the solution with initial conditions is

$$y(x) = -e^{-x} + \frac{1}{2}xe^{-x} + \frac{1}{2}x^{-1}e^{-x}.$$

Step 4.
By L'Hospital's rule, we know $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} e^{-x} = 0$. (1 point)
We have (1 point)

$$\lim_{x\to\infty} y(x) = \lim_{x\to\infty} -e^{-x} + \frac{1}{2}\lim_{x\to\infty} xe^{-x} + \frac{1}{2}\lim_{x\to\infty} x^{-1}e^{-x} = 0 + 0 + 0 = 0.$$