1021微乙01-05班期中考解答和評分標準

1. (25%) 設 $y = \frac{x^2}{2} + \frac{1}{x}$。回答下列問題，填入空格並給出**理由**(含計算)。若所求該項**不存在**，請填"**無**"。
(a) 函數遞升之區間為______________, 遞降之區間為______________(共 6%)。
極大點為 $(x, y) =$ ______________(2%), 極小點為$(x, y) =$ ______________(2%)。
(b) 函數凹向上之區間為______________, 函數凹向下之區間為______________(共 6%)。
反曲點為 $(x, y) =$ ______________(2%)。
(c) 漸近線為______________(3%)。
(d) 畫函數圖，標明升降處、上凹下凹處、極大極小點、反曲點及漸近線。(若有的話)(4%)

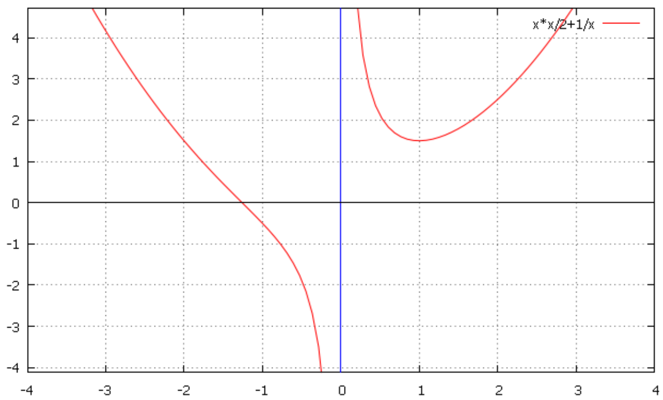
Solution:

(a) $y' = x - \frac{1}{x^2}$ (2 point)
$y' > 0$ iff $(x-1)(x^2+x+1) > 0$ iff $x > 1$, so the increasing interval is $(1, \infty)$ (2 point)
similarly the decreasing interval is $(-\infty, 1)$ except 0 (2 point)
there is no local maximum (2 point)
the local minimum is $(1, \frac{3}{2})$ (2 point)

(b) $y'' = 1 + \frac{2}{x^3}$ (2 point)
$y'' > 0$ iff $(x^3+2)x > 0$ iff $x > 0$ or $x < -\sqrt[3]{2}$ (2 point)
similarly the interval of concave down is $(-\sqrt[3]{2}, 0)$ (2 point)
the inflection point is $(-\sqrt[3]{2}, 0)$ (2 point)

(c) asymptote line: $x = 0$ (or $x = 0$, $y = \frac{x^2}{2}$) (2pt)
Reason (1 pt)
Since $\lim_{|x|\to\infty} y - \frac{x^2}{2} = \lim_{|x|\to\infty} 1/x = 0$, the asymptote curve is $\frac{x^2}{2}$. We know that there is no slope or horizontal asymptote. And $\lim_{x\to\pm 0} y = \pm\infty$ which implies there is a vertical asymptote at $x = 0$ (no other horizontal asymptote since y is continuous for $x \neq 0$.)

(d) I use gnu plot(a free software) to plot. (1pt respectively, in principle)



local min: $(1, 3/2)$, no local max
inflection pt: $(-2^{1/3}, 0)$
asymptote line: $x = 0$
increasing: $x > 1$, decreasing: $x < 1$, $x \neq 0$
convex upward: $x > 0$ or $x < -2^{1/3}$, convex downward: $-2^{1/3} < x < 0$

2. (15%) 令 $y = f(x)$ 滿足 $\tan^{-1}\frac{y}{x} + \ln\frac{x^2+y^2}{2} = \frac{\pi}{4}$。用 x, y 表出 y' 及 y''。並求出 y' 及 y'' 在點 $(x,y) = (1,1)$ 之值。

Solution:

$\arctan\frac{y}{x} \xrightarrow[\text{by chain rule}]{\text{differentate}} \frac{1}{1+(\frac{y}{x})^2} \times \frac{y'x-y}{x^2}$ [3pts]

$ln\frac{x^2+y^2}{2} \xrightarrow[\text{by chain rule}]{\text{differentate}} \frac{2}{x^2+y^2} \times \frac{2x+2yy'}{2}$ [2pts]

$\arctan\frac{y}{x} + ln\frac{x^2+y^2}{2} = \frac{\pi}{2} \Rightarrow \frac{1}{1+(\frac{y}{x})^2} \times \frac{y'x-y}{x^2} + \frac{2}{x^2+y^2} \times \frac{2x+2yy'}{2} = 0$ [2pts]

$y' = \frac{y-2x}{x+2y} \xrightarrow{x=1,y=1} y' = \frac{-1}{3}$ [2pts]

By Differentating agian:

$y'' = \frac{(y'-2)(x+2y)-(1+2y')(y-2x)}{(x+2y)^2}$ [4pts]

$y'' = \frac{5xy'-5y}{(x+2y)^2} \xrightarrow{x=1,y=1,y'=\frac{-1}{3}} y'' = -\frac{20}{27}$ [2pts]

3. (12%) 求 $y = \frac{\cos x}{1+\sin x}$ 在 $x = \frac{\pi}{6}$ 之切線。

Solution:

$$f(x) = \frac{\cos x}{1+\sin x}\Big|_{x=\frac{\pi}{6}} = \frac{\sqrt{3}}{3} \qquad \text{(2 points)}$$

$$f'(x) = \frac{-\sin x - \sin^2 x - \cos^2 x}{(1+\sin x)^2}\Big|_{x=\frac{\pi}{6}} = \frac{-1}{1+\sin x}\Big|_{x=\frac{\pi}{6}} = -\frac{2}{3} \qquad \text{(7 points)}$$

$$y - f(\frac{\pi}{6}) = f'(\frac{\pi}{6})\left(x - \frac{\pi}{6}\right)$$
$$y - \frac{\sqrt{3}}{3} = -\frac{2}{3}\left(x - \frac{\pi}{6}\right)$$

(2+1 points)

4. (12%) 求 $\lim_{x\to 0}\frac{\cos x - 1}{x\sin x}$。

Solution:

這裡給出兩種主要做法與評分標準:

Solution 1: 上下同乘以$(\cos x+1)$, 我們得到

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x-1}{x\sin x} &= \lim_{x\to 0}\frac{\cos^2 x-1}{x\sin x(\cos x+1)} \quad (3\%)\\ &= \lim_{x\to 0}\frac{-\sin^2 x}{x\sin x(\cos x+1)}(1\%)\\ &= \lim_{x\to 0}\frac{-\sin x}{x(\cos x+1)}\\ &= -\lim_{x\to 0}\frac{\sin x}{x}\cdot\frac{1}{\cos x+1}(2\%)\\ &= -1\cdot\frac{1}{2}(\text{每個極限各}3\%)\\ &= \frac{-1}{2}\end{aligned}$$

Solution 2: 根據倍角公式$\cos x = 1-2\sin^2\frac{x}{2}$ 以及 $\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2}$ (共2%), 我們得到

$$\begin{aligned}\lim_{x\to 0}\frac{\cos x-1}{x\sin x} &= \lim_{x\to 0}\frac{(1-2\sin^2\frac{x}{2})-1}{x\sin x} \quad (3\%)\\ &= \lim_{x\to 0}\frac{-2\sin^2(\frac{x}{2})}{x(2\sin\frac{x}{2}\cos\frac{x}{2})} \quad (3\%)\\ &= -\lim_{x\to 0}\frac{\sin\frac{x}{2}}{\frac{x}{2}}\cdot\frac{1}{2}(1\%)\\ &= -1\cdot\frac{1}{2}(3\%)\\ &= \frac{-1}{2}\end{aligned}$$

使用羅畢達法則(L'Höpital rule)而沒有給出正確證明的同學最高僅會獲得一半的分數(6分). 若先化簡才使用羅畢達法則的話,會獲得全部化簡的分數然後得到剩下分數的一半(無條件捨去).

5. (12%) 用線性逼近求 $(128)^{\frac{1}{3}}$ 之近似值。

Solution:
線性逼近 $(128)^{\frac{1}{3}}$
令 $f(x)=x^{\frac{1}{3}}$， 則$f'(x)=\frac{1}{3}x^{-\frac{2}{3}}$

$$f(x)=f(x_0)+f'(x_0)(x-x_0) \quad (4\ pts)$$

取 $x_0=125$， 所以

$$\begin{aligned}(128)^{\frac{1}{3}}=f(128) &= f(125)+f'(125)(128-125) \quad (7\ pts)\\ &= 5+\frac{1}{3}\times\frac{1}{25}\times 3 \quad (10\ pts)\\ &= 5+\frac{1}{25}\\ &= \frac{126}{25} \quad (12\ pts)\\ &= 5.04\end{aligned}$$

6. (12%) 令 $g(x)$ 為 $f(x) = x^5 + 2x^3 + x - 2$ 之反函數，求 $g'(f(1))$ 之值。

Solution:

$$g'(f(1)) = \frac{1}{f'(1)} = \frac{1}{12}$$

Grading
If student has the concept of inverse function and differential but use it in wrong way or gives nothing then gives 2 to 4 point
If student gives relation of the differential of inverse function but use it something wrong gives 6 to 8 point
If only calculation errors gives 11 point

7. (12%) 說明 $y = x$ 與 $y = \tan^{-1} x$ 只有一個交點。

Solution:
Let $f(x) = x - \tan^{-1} x$.
It is clear that $f(0) = 0 - \tan^{-1} 0 = 0$.
Now suppose that there is another $k \neq 0$, such that $f(k) = 0$. (We want to get a contradiction.)
Then we have $f(0) = 0 = f(k)$.
By Rolle's theorem, there is a between 0 and k (not contain 0 and k), such that $f'(a) = 0$.
Then

$$\begin{aligned} & 0 = f'(a) = (1 - \frac{1}{1+x^2})|_{x=a} = \frac{a^2}{1+a^2} \\ \Rightarrow \quad & 0 = a^2 \\ \Rightarrow \quad & a = 0 \end{aligned}$$

However, a can not be equal to 0, so the assupmtion must be wrong.
Hence, only when x is equal to 0, $f(x) = 0$.
Then we have proved that there is only one intersection point.

scoring standard:
(1) If you have written down the derivative $(\tan^{-1})' = \dfrac{1}{1+x^2}$, you will get 4 points.
(2) If you have written down the complete and correct statement of MVT or Rolle's theorem, you will get 4 points.
(3) If you can derive a contradiction form your assumption, you will get 4 points.