1021微乙01-05班期末考解答

1. (15%) 令 $I_n = \int (\ln x)^n dx,\ n \geq 1$

   (a) (6%) 求 $I_1 =?$

   (b) (6%) 把 I_{n+1} 用 I_n 表出。

   (c) (3%) 利用(b)的遞迴式求 I_4。

Solution:

令 $I_n = \int (\ln x)^n \, dx,\ n \geq 1.$

(a) 使用分部積分法, 令$u = \ln x,\ dv = dx$ (2%), 我們得到

$$\begin{aligned}\int \ln x \, dx &= x \ln x - \int x \, d(\ln x) \quad (1\%) \\ &= x \ln x - \int x \cdot \frac{1}{x} \, dx \quad (1\%) \\ &= x \ln x - x + C \quad (2\%)\end{aligned}$$

註：未加上積分常數扣1分, 沒有計算過程根據規則得0分.

(b) 使用分部積分法, 令$u = (\ln x)^{n+1},\ dv = dx$ (2%), 我們得到

$$\begin{aligned}I_{n+1} = \int (\ln x)^{n+1} \, dx &= x(\ln x)^{n+1} - \int x \, d(\ln x)^{n+1} \quad (1\%) \\ &= x \ln x - \int x \cdot (n+1)(\ln x)^n \cdot \frac{1}{x} \, dx \quad (2\%) \\ &= x(\ln x)^{n+1} - (n+1) I_n \quad (1\%)\end{aligned}$$

註1：如果僅計算前幾個, 如I_2, I_3, 然後直接推論猜測規律出最後結果的人, 得到4分.

註2：使用別的方法計算出來的遞迴式也會有分數, 上述只是其中一個較好計算的結果.

(c) 根據遞迴式我們得到

$$\begin{aligned}I_4 &= x(\ln x)^4 - 4I_3 \\ &= x(\ln x)^4 - 4\left[x(\ln x)^3 - 3I_2\right] \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12\left[x(\ln x)^2 - 2I_1\right] \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24(x \ln x - x + C) \\ &= x(\ln x)^4 - 4x(\ln x)^3 + 12x(\ln x)^2 - 24x \ln x + 24x + C\end{aligned}$$

評分方法：

(i) (b) 若計算錯, 則(c)得0分.

(ii) 計算錯誤, 如正負號錯誤, 括號有誤等, 或是未加上積分常數總和至多扣1分.

(iii) 未使用(b)小題中所得到的遞迴式的人得1分.

2. (12%) (a) (6%) 求 $\int \frac{x+1}{x^2+x+1}dx$。

(b) (6%) 求 $\int \frac{dx}{e^x(e^{2x}-1)}$。

Solution:

(a)

$$\begin{aligned} & \int \frac{x+1}{x^2+x+1}\ dx \\ = & \int \frac{x+\frac{1}{2}}{x^2+x+1}\ dx + \frac{1}{2}\int \frac{dx}{x^2+x+1} \quad (2\%) \\ = & \frac{1}{2}\ln(x^2+x+1) + \frac{1}{\sqrt{3}}\tan^{-1}(\frac{2}{\sqrt{3}}(x+\frac{1}{2})) + C \quad (4\%) \end{aligned}$$

在答案接近完整時，沒加 C 扣一分，但不重複扣!

(b)

$$\begin{aligned} & \int \frac{dx}{e^x(e^{2x}-1)}, \quad \text{令} \quad u = e^x \quad (2\%) \\ = & \int (\frac{1}{2}\frac{1}{u-1} - \frac{1}{2}\frac{1}{u+1} - \frac{1}{u^2})\ du \quad (2\%) \\ = & \frac{1}{2}\ln|u-1| - \frac{1}{2}\ln|u+1| + \frac{1}{u} + C \\ = & \frac{1}{2}\ln|e^x-1| - \frac{1}{2}\ln|e^x+1| + e^{-x} + C \quad (2\%) \end{aligned}$$

3. (12%) (a) (6%) 求 $\int_0^{\frac{1}{2}} x^2\sqrt{1-x^2}dx$。

(b) (6%) 求 $\frac{d}{dx}\int_x^{x^2} \frac{dt}{1+t^5}$。

Solution:

(a) [6pts]

$$
\begin{aligned}
&\int_0^{\frac{1}{2}} x^2\sqrt{1-x^2}dx \\
\xrightarrow[x=\sin\theta]{dx=\cos\theta}\ &\int_0^{\frac{\pi}{6}} \sin^2\theta\cos^2\theta d\theta \quad [2pts] \\
\xrightarrow[\cos^2\theta=\frac{1+\cos 2\theta}{2}]{\sin^2\theta=\frac{1-\cos 2\theta}{2}}\ &\int_0^{\frac{\pi}{6}} \frac{1-\cos^2 2\theta}{4}d\theta \\
=&\int_0^{\frac{\pi}{6}} \frac{\sin^2 2\theta}{4}d\theta \\
\xrightarrow{\sin^2 2\theta=\frac{1-\cos 4\theta}{2}}\ &\int_0^{\frac{\pi}{6}} \frac{1-\cos 4\theta}{8}d\theta \quad [2pts] \\
=&\frac{\theta}{8}-\frac{1}{32}\sin 4\theta\Big|_0^{\frac{\pi}{6}} \\
=&\frac{\pi}{48}-\frac{\sqrt{3}}{64} \quad [2pts]
\end{aligned}
$$

(b) [6pts]

$$
\begin{aligned}
&\frac{d}{dx}\int_x^{x^2} \frac{dt}{1+t^5} \\
\xrightarrow[f(x)=\frac{1}{1+t^5}]{F(x)=\int_0^x f(t)dt}\ &\frac{d}{dx}\{F(x^2)-F(x)\} \quad [2pts] \\
=&\frac{dF(x^2)}{dx^2}\frac{dx^2}{dx}-\frac{dF(x)}{dx} \quad [2pts] \\
\xrightarrow{\text{Fundamental Calculus}}\ &f(x^2)\cdot 2x-f(x) \\
=&\frac{2x}{1+x^{10}}-\frac{1}{1+x^5} \quad [2pts]
\end{aligned}
$$

4. (12%) (a) (6%) 求 $\int \sec^3\theta d\theta$。(若知道 $\int \sec\theta d\theta$ 可以使用)

(b) (6%) 求曲線 $y = \frac{x^2}{2} + 1$ 由 $x = 0$ 到 $x = 2$ 之弧長。

Solution:

(a)

$$\begin{aligned}
\int sec^3\theta\, d\theta &= \int \underline{(1+tan^2\theta)}sec\theta\, d\theta \\
&= \int sec\theta\, d\theta + \int tan^2\theta sec\theta\, d\theta && (1\, points) \\
&= \int sec\theta\, d\theta + \int tan\theta\, dsec\theta \\
&= \int sec\theta\, d\theta + \underline{tan\theta sec\theta - \int sec^3\, d\theta} && (2\, points) \\
&= \underline{ln|sec\theta + tan\theta|} + tan\theta sec\theta - \int sec^3\, d\theta && (2\, points) \\
\Longrightarrow \int sec^3\theta\, d\theta &= \frac{1}{2}ln|sec\theta + tan\theta| + \frac{1}{2}tan\theta sec\theta + C && (1\, points)
\end{aligned}$$

or

$$\begin{aligned}
\int sec^3\theta\, d\theta &= \int sec^3\theta \frac{cos\theta}{cos\theta}\, d\theta \\
&= \int sec^4\theta\, dsin\theta \\
&= \int \frac{1}{(1-sin^2\theta)^2}\, dsin\theta \\
(Let\ u = sin\theta) \quad &= \int \frac{1}{(1-u^2)^2}\, du \\
&= \int \frac{1}{(1-u)^2(1+u)^2}\, du && (2\, points) \\
&= \frac{1}{4}\int \frac{1}{(1-u)} + \frac{1}{(1+u)} + \frac{1}{(1-u)^2} + \frac{1}{(1+u)^2}\, du && (2\, points) \\
&= \frac{1}{4}\left[-ln|1-u| + ln|1+u| + \frac{1}{(1-u)} - \frac{1}{(1+u)}\right] + C \\
&= \frac{1}{4}\left[ln\left|\frac{1+u}{1-u}\right| + \frac{2u}{1-u^2}\right] + C \\
&= \frac{1}{4}\left[ln\left|\frac{1+sin\theta}{1-sin\theta}\right| + \frac{2sin\theta}{1-sin^2\theta}\right] + C && (2\, points) \\
&= \frac{1}{4}ln\left|\frac{(1+sin\theta)^2}{1-sin^2\theta}\right| + \frac{1}{2}\frac{sin\theta}{cos^2\theta} + C \\
&= \frac{1}{4}ln\left|\frac{1+sin\theta}{cos\theta}\right|^2 + \frac{1}{2}tan\theta sec\theta + C \\
&= \frac{1}{2}ln\left|tan\theta + sec\theta\right| + \frac{1}{2}tan\theta sec\theta + C
\end{aligned}$$

(b)

$$\int_0^2 \sqrt{1+(y')^2}\,dx = \int_0^2 \sqrt{1+x^2}\,dx \qquad (2\,points)$$

$$(Let\ x = tan\theta) \qquad = \int_{tan^{-1}0}^{tan^{-1}2} \sqrt{1+tan^2\theta}\,dtan\theta$$

$$= \int_{tan^{-1}0}^{tan^{-1}2} sec^3\theta\,d\theta \qquad (2\,points)$$

$$= \left[\frac{1}{2}ln\,|tan\theta + sec\theta| + \frac{1}{2}tan\theta sec\theta\right]_{tan^{-1}0}^{tan^{-1}2}$$

$$= \frac{1}{2}ln|\sqrt{5}+2| + \sqrt{5} \qquad (2\,points)$$

5. (10%) 求由 $x=0$, $x=1$, $y=0$, $y=\sqrt{x^2+1}$ 所圍區域對 y- 軸旋轉的旋轉體體積.

Solution:

Method I

$$\begin{aligned} & \int_0^1 2\pi x\sqrt{x^2+1}\ dx \quad (5\%) \\ = & \ \frac{2}{3}\pi(x^2+1)^{\frac{3}{2}}\Big|_0^1 \quad (3\%) \\ = & \ \frac{(4\sqrt{2}-2)\pi}{3} \quad (2\%) \end{aligned}$$

Method II

$$\begin{aligned} & \int_1^{\sqrt{2}} \pi(1^2-(y^2-1))\ dy + \int_0^1 \pi\ dy \\ = & \ \frac{(4\sqrt{2}-2)\pi}{3} \end{aligned}$$

6. (15%)

(a) (8%) 寫出 $f(x) = \sin x$ 在 $x = 0$ 處之第四次泰勒多項式，並寫出泰勒定理中之餘項。

(b) (7%) 求 $\sin 20^\circ$ 之近似值，誤差小於 10^{-4}。答案可用 π 表示，不必帶入 $\pi = 3.14\cdots$，餘項需仔細估計，以證明近似值之準確度。

Solution:

(a)
$f(x) = x - x^3/3! + cos(c)x^5/5!$, form some c

(b)
$sin(\pi/9) = (\pi/9) - (\pi/9)^3/3! + cos(c)(\pi/9)^5/5!$ (3 points)
$|R(\pi/9)| \le (\pi/9)^5/120$, where $R(\pi/9)$ means the remainder term (2 points)
compute $(\pi/9)^5/120 \le (2/5)^5/120 \le 4/46875 \le 1/10000$ (2 points)

7. (12%) (a) (6%) 求 $\lim_{x\to\infty} x^{\frac{1}{x}}$。

(b) (6%) 求 $\lim_{x\to 0} \dfrac{\ln(1+x^2)}{1-\cos x}$。

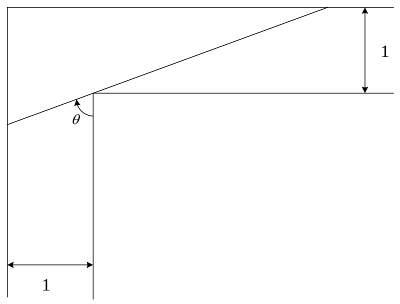
Solution:

(1)$lim_{x->\infty} x^{\frac{1}{x}} = lim_{x->\infty} \exp(\frac{1}{x}\ln(x))(3pt)$ If has some calculation error 2pt

$= lim_{x->\infty} \exp(\frac{1}{x})(LHrule) = 1(3pt)$

(2) $lim_{x->0}\frac{\ln(1+x^2)}{1-\cos(x)}$=$lim_{x->0}\frac{\frac{2x}{1+x^2}}{sin(x)}$(LH rule) (3PT) =$lim_{x->0}\frac{\frac{2}{1+x^2}}{\frac{sin(x)}{x}}$ =2 (3PT)

8. (12%) 一人欲扛著長竹竿，要維持水平通過下圖之窄巷(寬為 1, 1 公尺)，請問竹竿之長度最長可以幾公尺?



Solution:

Observe that the length must be not larger than $f(\theta) := 1/\sin\theta + 1/\cos\theta$, so it suffices to find the min. of f. (3pt)

$f'(\theta) = \sin\theta/\cos^2\theta - \cos\theta/\sin^2\theta = \dfrac{\sin^3\theta - cos^3\theta}{\sin^2\theta\cos^2\cos^2\theta}$, so f'=0 at $\theta = \pi/4$ (3pt); f'$>$ 0 for $\pi/4 < \theta < \pi/2$ and f'$<$ 0 for $0 < \theta < \pi/4$ (3pt).

So length $\leq f(\pi/4) = 2\sqrt{2} \leq f(\theta)$ (3pt)